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Spinors on manifolds with boundary: APS index theorems with torsion

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Abstract: Index theorems for the Dirac operator allow one to study spinors on manifolds with boundary and torsion. We analyse the modifications of the boundary Chern-Simons correction and APS η invariant in the presence of torsion. The bulk contribution must also be modified and is computed using a supersymmetric quantum mechanics representation. Here we find agreement with existing results which employed heat kernel and Pauli-Villars techniques. Nonetheless, this computation also provides a stringent check of the Feynman rules of de Boer et al. for the computation of quantum mechanical path integrals. Our results can be verified via a duality relation between manifolds admitting a Killing-Yano tensor and manifolds with torsion. As an explicit example, we compute the indices of Taub-NUT and its dual constructed using this method and find agreement for any finite radius to the boundary. We also suggest a resolution to the problematic appearance of the Nieh-Yan invariant multiplied by the regulator $(\text{mass})^2$ in computations of the chiral gravitational anomaly coupled to torsion.

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Contents

1. Introduction

Index theorems for manifolds with boundary and torsion involve a detailed understanding of spinors on such manifolds. In the torsion free case this problem was understood long ago in the mathematics literature and is solved by the Atiyah-

Patodi-Singer index theorem [1]. Essentially, they found that the index for manifolds with boundary is the sum of the usual bulk result for the index (proportional to the integral over the first Pontrjagin class) plus a correction from the so-called η invariant of the boundary Dirac operator. Upon the addition of torsion, further questions arise such as how does torsion affect the computation of the index in the bulk? What are the appropriate boundary conditions for spinors on such manifolds and how does one define the adjoint of operators acting on these spinors? How does torsion modify the computation of the η invariant? Are there corrections to the index over and above that found by Gilkey [2] if the metric is not of product form near the boundary? From a physical viewpoint the resolution of these questions is of considerable interest, not only because of the relation between the index of the Dirac operator and anomalies, but also due to the considerable importance attached to the physics of supergravity theories (involving, of course, both spinors and torsion) on manifolds with boundary, e.g. anti de Sitter spaces [3].

Manifolds in the Einstein-Cartan theory which include torsion are not nearly as well studied as those without, and solutions of the equations of motion tend to be of considerable complexity, making explicit checks of the general formalism we are about to present cumbersome. Fortunately, we can relate the index of a manifold $\mathcal{M}$ with torsion to a torsion free manifold $\widetilde{\mathcal{M}}$ when their Dirac operators, $\not{D}(e,A)$ and $\not{D}(\tilde{e},\widetilde{V})$ respectively, satisfy[1]

$$\{\,\not{D}(e,A),\not{D}(\tilde{e},\widetilde{V})\,\}\;=\;0 \tag{1.1}$$

(note that $\not{D}(\tilde{e},\widetilde{V})$ denotes the Dirac operator with inverse vierbein $\tilde{e}_r{}^\mu$ coupled minimally to an abelian vector field $\widetilde{V}^\mu$, whose necessity will be explained later, and $\not{D}(e,A)$ refers to inverse vierbein $e_r{}^\mu$ and coupling to totally antisymmetric torsion which may be re-expressed via Hodge duality as minimal coupling to an axial vector field A^μ). This equation will be made mathematically more precise later, it suffices to say here that if we are able to find a pair of manifolds whose Dirac operators are related in this way, we will be able to test our results for index theorems on manifolds with boundary in a concrete example. In what follows we will refer to the manifold with torsion $\mathcal{M}$ as the "dual" of the torsion-less manifold $\widetilde{\mathcal{M}}$.

In fact, as was pointed out in [5], for the case where one takes the manifold $\widetilde{\mathcal{M}}$ as the background for classical spinning particle models with extended supersymmetry, one may view $\not{D}(\tilde{e},\widetilde{V})=\gamma^r\widetilde{e}_r{}^\mu\partial_\mu+\cdots$ as a supersymmetry generator and then search for additional supersymmetry generators $\not{D}(e,A)=\gamma^r e_r{}^\mu\partial_\mu+\cdots$. The tensor $e_r{}^\mu$ is known as a Killing-Yano tensor and may be viewed as the inverse vierbein of the dual manifold $\mathcal{M}$. We also stress that in the quantum case, when one considers

[1]The existence of a relation of this type was first suggested by Jan-Willem van Holten based on the vanishing classical Poisson brackets between the supercharges of manifolds with Killing-Yano supersymmetries [4].

the Dirac operators $\not{D}(e,A)$ and $\not{D}(\tilde{e},\widetilde{V})$ depending on Dirac matrices satisfying a Clifford algebra, equation (1.1) is more subtle than in the classical case where one has supersymmetry charges depending on world line Majorana spinors obeying canonical Poisson brackets. The quantisation of Killing-Yano symmetries is, in fact, an important aspect of our work.

At the end of this paper we take as an example $\widetilde{\mathcal{M}}$ to be Taub-NUT,

$$d\tilde{s}^2 = \frac{r+2m}{r}\left[\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right] + \frac{4rm^2}{r+2m}\left[\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right]^2, \tag{1.2}$$

a manifold of considerable physical interest and $\mathcal{M}$ is then the dual Taub-NUT manifold with metric

$$\mathrm{d}s^2 = \frac{r+2m}{r}\left[\mathrm{d}r^2 + \frac{m^2r^2}{(r+m)^2}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] + \frac{4rm^2}{r+2m}\left[\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right]^2, \tag{1.3}$$

along with totally antisymmetric torsion (our normalisations can be found in the appendix)

$$T = -\frac{4r^2m^2\sin\theta}{(r+m)^2}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi\,. \tag{1.4}$$

We should mention that the dual Taub-NUT manifold possesses neither self-dual Riemann curvature tensors nor satisfies the Einstein-Cartan equations but this is unimportant for our purposes since it does satisfy the central equation (1.1).

Our chief line of argument to compute the index of a manifold with torsion and boundary follows the Atiyah-Patodi-Singer index theorem [1]. We split the computation into a contribution from the bulk and a boundary contribution. The bulk contribution is computed by representing the index as the regulated trace of γ^5

$$\mathrm{index}\,\not{D} \;=\; \mathrm{Tr}\left[\gamma^5\,\exp\left(\frac{\beta}{2}\not{D}^{\,2}\right)\right], \tag{1.5}$$

which in turn may be represented as a supersymmetric quantum mechanical path integral [6]. In this approach, however, one essentially inserts infinitely many sets of plane wave resolutions of unity, for which one may clearly integrate by parts. Therefore this calculation is insensitive to the precise boundary conditions obeyed by the states appearing in the trace in (1.5) and yields only the bulk contribution to the index. In fact, analogous computations have also been carried out in a heat kernel [7] as well as Pauli Villars [8] framework and yield results identical to ours. Nonetheless, our independent computation is interesting in its own right since it provides a stringent test of the rigorous quantum mechanical path integral formalism for sigma models in curved space developed by de Boer et al. [9, 10] (and in particular of the Christ and Lee $\hbar^2$ counter terms [11] that must be added to the action in order that the path integral provides a faithful representation of the trace (1.5)). The details of this computation are described in section 2.

Having calculated the bulk contribution one must compute the boundary contribution which has two parts. The first is the η invariant for the boundary manifold. The η invariant is a spectral invariant of the boundary Dirac operator and may be computed from the eigenvalues of this operator. In the torsion free case this problem was solved for various asymptotic geometries by Hitchin (and applied to the Taub-NUT manifold in [12, 13, 14]). Therefore, we generalise the computation of Hitchin to include the dual Taub-NUT example. To this end we construct and solve the boundary Dirac operator in the presence of torsion which allows us to compute the η invariant for this manifold.

The second boundary contribution, due to Gilkey [2], is necessary in the case where the manifold is not of product form at the boundary. The main idea is that the APS analysis is valid for manifolds which approach a cylinder whose cross section is the boundary manifold. If this is not the case, one must make an extra boundary correction proportional to the integral over the boundary of the difference between the Chern-Simons form computed using the product metric and that computed using the metric itself. This computation can be easily generalised to the torsion-full case where we find a boundary correction proportional to a generalised Chern-Simons term. Note, in particular, that all our results hold for manifolds with boundaries at any finite radius, rather than just in some non-compact limit. The derivations of the boundary corrections are to be found in section 3.

Section 4 analyses in detail the precise relation between index theorems of manifolds possessing mutually anticommuting Dirac operators. This relation requires a careful understanding of boundary conditions for spinors and therefore the space of states in the trace (1.5). Subsection 4.2 outlines the main ingredients of the Killing-Yano technique to generate manifolds with mutually anticommuting Dirac operators and the explicit geometric data for our example manifolds, Taub-NUT and dual Taub-NUT may be found in 4.3. Also included in subsection 4.2 is the generalisation of the Killing-Yano construction to the quantum case.

Urgent readers not wishing to wade through all the details may find the final expression for the index with torsion on manifolds with boundary in equation (3.31). A summary of the results we obtained for the example manifolds is presented in subsection 4.4. Finally, more speculative remarks along with possible applications of our results are reserved for the conclusion where, in particular, we discuss the controversial $1/\beta$ regulator $(\text{mass})^2$ Nieh-Yan contribution to the index in light of our work. In passing, we note that many of the issues we encounter have been dealt with in some depth in the mathematics and physics literature so that we have attempted to present these more formal aspects in an accessible format.

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2. Bulk contribution to the index with torsion

2.1 The Dirac index as a quantum mechanical path integral

This section is devoted to the computation of the bulk computation to the index employing the supersymmetric path integral approach of Alvarez-Gaumé and Witten [6]. The bulk contribution to the index of the Dirac operator with torsion has already been computed in [7] via a heat kernel expansion and in [8] using Pauli-Villars regularisation, needless to say, our results exactly reproduce those of these authors. Besides the high degree of certainty one now has in the result (see equation (2.19)), our motivations for performing this calculation were fourfold:

1. Our computation is carried out in the framework of rigorously defined quantum mechanical curved space path integrals introduced by de Boer et al. [9, 10] and therefore provides a stringent check of their work. In particular, we note that the result depends crucially on the Christ and Lee terms [11] (see the second line of (2.13)) that one must add to the action appearing in the path integral at order $\hbar^2 = \beta^2$ in order to employ propagators derived via a midpoint rule.

2. There has been some controversy in the literature over the appearance of a term in the (bulk) index with torsion that diverges as one takes the limit in which the regulator is removed [15, 16, 17, 18, 19, 20]. This term is proportional to the Nieh-Yan tensor (see the first term on the right hand side of (2.19)) and arises at tree level in our path integral expansion. In the path integral approach, one clearly sees the importance of including this term in order to obtain the correct results for the bulk index (i.e. the regulator independent contribution) since it appears as a vertex as well as in disconnected graphs.

3. The path integral derivation is based on inserting infinitely many plane wave resolutions of unity. Therefore one clearly sees that it is only sensitive to the bulk contribution, since differentiations by parts then ignore any boundary terms.

4. The quantum mechanical model itself is of intrinsic interest since it describes a spinning particle moving through a background with torsion. Obviously, the construction of the precise path integral quantisation of this model is an important step in understanding its quantum dynamics.

The index of the Dirac operator $\not{D}$ is given by the trace over the spectrum

$$\text{index}\,\not{D} = \mathrm{Tr}\left[\gamma^5 \exp\left(\frac{\beta}{2}\not{D}^{\,2}\right)\right]. \tag{2.1}$$

When there is a gap between the zero-modes and the remainder of the spectrum [21] this result should, of course, be β-independent. However, for manifolds with boundary, one must carefully take into account the states traced over. If one is interested in the bulk contribution only, the trace can be evaluated by inserting N sets of plane waves (where $\mathrm{d}t \sim 1/N \to 0$) and the result may then be represented as a path integral. The expression one then finds is an expansion in powers of β

$$\beta^{-1}\Delta_{-1}(\not{D}\,) + \Delta_0(\not{D}\,) + O(\beta) \equiv \Delta(\not{D}\,)\,. \tag{2.2}$$

The one-loop beta independent contribution $\Delta_0(\not{D}\,)$ yields the bulk result for the index and as will be explained in section 3, the complete result for the index is obtained by including the APS and Chern-Simons boundary correction terms. Here we will compute the terms $\Delta_{-1}(\not{D}\,)$ and $\Delta_0(\not{D}\,)$. We comment on the interpretation of the $1/\beta$ contribution $\Delta_{-1}(\not{D}\,)$ in the conclusion.

Before introducing the details of our path integral computation, let us briefly summarise the main ingredients of the approach. To begin with, one treats the exponential in the trace (2.1) as the imaginary time evolution operator of a quantum mechanical system. The *operator* in the exponent corresponds to the hamiltonian. Importantly, its operator ordering is now fixed so that the usual step where one begins with some classical hamiltonian generating dynamics via Poisson brackets and then elevates this expression to an operator, fixing the ordering by symmetry principles (or ultimately experiment!) is not needed. The operator $H = -\beta^2 \not{D}\,^2/2$ *is* the *quantum* hamiltonian.

The next step is to derive the path integral by inserting N complete sets of states. The achievement of de Boer et al. was to derive a path integral which faithfully represents the operator expression (2.1). The point being that once one takes the continuum limit $N \to \infty$, the discretised expressions derived at finite N become distributions whose products (which appear in Feynman diagrams) are ambiguous. However, given the rigorous discretised expression for the path integral it is always possible to resolve these ambiguities in a unique and finite fashion. Their derivation was performed by *rewriting* the quantum hamiltonian in a Weyl ordered form which allows infinitesimal transition elements at intermediate steps in the path to be evaluated by a midpoint rule which leads to a certain set of vertices, propagators and rules for products of distributions. Of course another ordering principle would have yielded different propagators and vertices, the key point is to derive a path integral which precisely represents[2] (2.1).

[2]It is interesting to compare this situation to what one finds in quantum field theory. In quantum field theory, one must regulate and renormalise to obtain a path integral yielding finite, well defined results for the products of distributions appearing at loop level in Feynman diagrams. Furthermore, one makes some choice of renormalisation point in order to make contact with physical quantities. Of course, in quantum mechanics, there is no anomaly in scale invariance and the path integral $Z[y^\mu, \Psi^r]$ (see (2.6)) should provide an unambiguous and finite representation of the transition amplitude $\langle y, \Psi|\,\gamma^5 \exp \frac{\beta}{2}\not{D}\,^2\,|y,\Psi\rangle$. Although, the quantum mechanical hamiltonian $H = -\beta^2\not{D}\,^2/2$ has a definite operator ordering, in passing to a path integral representation, the vertices of the path integral are ambiguous up to reorderings of the hamiltonian. However, once a particular or-

We now turn to the details our calculation and begin by thinking of the exponential in (2.1)

$$\exp\left[-i\left(-\frac{\beta^2 \not{D}^2}{2}\right)\left(\frac{-i}{\beta}\right)\right] \tag{2.3}$$

as a quantum mechanical evolution operator $\exp[-iHt/\hbar]$ with hamiltonian $\frac{-\beta^2\not{D}^2}{2}$, unit Euclidean time interval and Planck's constant $\hbar = \beta$. The Dirac matrices are then identified with fermionic coordinates and the derivative operator ∂_μ with the canonical momentum of the bosonic coordinate x^μ,

$$\begin{aligned} \gamma^r &= \sqrt{2}\,\beta^{-1/2}\,\psi^r\,, & \partial_\mu &= \beta^{-1} g^{\frac{1}{4}}\, i p_\mu\, g^{-\frac{1}{4}}\,, \\ \{\psi^r, \psi^s\} &= \beta\,\delta^{rs}\,, & [x^\mu, p_\nu] &= i\beta\,\delta^\mu{}_\nu\,. \end{aligned} \tag{2.4}$$

The similarity transformation for the canonical momentum p_μ (with $g = \det g_{\mu\nu}$) implies the inner product $\langle x|y\rangle = g^{-1/2}\delta^4(x-y)$. Observe also that the rescalings by appropriate factors of β in (2.4) ensure that β plays the rôle of Planck's constant in the canonical commutation relations and that the hamiltonian $-\beta^2\not{D}^2/2$ begins with a "classical" β independent contribution, plus possibly terms linear and quadratic in β (depending on the precise ordering in which one writes the operators x^μ, p_μ and ψ^r).

We may now represent the index (2.1) by a path integral with periodic boundary conditions, the presence of γ^5 in the trace yields periodic boundary conditions for the fermions too:

$$\Delta(\not{D}) = \int_{\mathcal{M}} \mathrm{d}^4 y\, \frac{\sqrt{g(y)}}{(2\pi i)^2}\, \mathrm{d}^4\Psi\, Z[y^\mu, \Psi^r]\,, \tag{2.5}$$

where the path integral Z is a function of constant real background fields y^μ and Ψ^r. Schematically

$$\begin{aligned} Z[y^\mu, \Psi^r] = \int &[\mathrm{d}q^\mu\, \mathrm{d}\eta^r\, \mathrm{d}a^\mu\, \mathrm{d}b^\mu\, \mathrm{d}c^\mu] \times \\ &\times \exp\left\{-\frac{1}{\beta}\int_0^1 \mathrm{d}t\, \mathcal{L}(x^\mu(t) = y^\mu + q^\mu(t), \psi^r(t) = \Psi^r + \eta^r(t))\right\}. \end{aligned} \tag{2.6}$$

As discussed above, one usually expects path integrals to be plagued with ambiguities associated with the normalisation and precise definition of the measure along with products of distributions in loops. Fortunately however, finite interval path integrals

dering (there is nothing sacrosanct about Weyl ordering, it is merely convenient) is fixed one must decide (derive) which rule is used to define products of distributions. In analogy with quantum field theory theory, the discretisation procedure and rule for the products of delta and Heaviside functions above may be thought of as regulating and renormalising the theory with renormalisation point judiciously chosen to reproduce the required transition amplitude with the given hamiltonian.

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in curved space have been studied in detail by de Boer et al. [9, 10]. In particular they have found the precise definition of Z above, i.e. the exact vertices and Feynman rules, such that the path integral representation (2.5) is identically equal to the index $\Delta(\not{D})$. A two loop verification of their work in the torsion-less case may be found in [22] along with the conventions employed in this paper. We now spell out the key details of this approach.

Firstly, although manifestly the quantum mechanical trace (2.1) is finite, in the path integral, closed bose loops $\langle \dot{q}^\mu(t)\dot{q}^\nu(t)\rangle$ yield delta function divergences $\delta(0)$ (first observed by Lee and Yang [23]). However, the decomposition of unity $1 = \int \mathrm{d}^4x\, g^{1/2}|x\rangle\,\langle x|$ yields a measure factor $g^{1/2}$ at each point of the path which one may exponentiate via bosonic and fermionic ghosts $a^\mu(t)$ and $\{b^\mu(t), c^\mu(t)\}$, respectively [24, 25] yielding the ghost action

$$\int_0^1 \mathrm{d}t\,\mathcal{L}_{\mathrm{ghost}} = \int_0^1 \mathrm{d}t\,\frac{1}{2}g(x(t))_{\mu\nu}\big(a(t)^\mu a(t)^\nu + 2\,b(t)^\mu c(t)^\nu\big)\,, \tag{2.7}$$

whose net effect is to precisely cancel the divergence $\delta(0)$ whenever it appears (for example graphs including a ghost loop are needed to cancel the divergence in the third graph of table 4).

Secondly, ambiguous products of distributions involving Heaviside and delta functions $\delta(t-s)\theta(t-s)$ also appear. The result of [9, 10] is that adopting vertices corresponding to a *Weyl ordered hamiltonian*, one finds propagators such that $\theta(0) = 1/2$ (strictly speaking, one returns to the discretised derivation of the path integral where the delta function is a Kronecker delta δ_{ij} and the propagators depend on the discrete Heaviside function with $\theta_{ii} = 1/2$). More precisely, one must *rewrite* the operator valued hamiltonian $H = -\beta^2\not{D}^2/2$ in Weyl ordered form. *Thereafter* one obtains the lagrangean[3]

$$\mathcal{L} = -ip_\mu\dot{x}^\mu + \frac{1}{2}\psi_r\dot{\psi}^r + H\,. \tag{2.8}$$

by replacing operators by c-numbers in the Weyl ordered hamiltonian. In general one then finds additional vertices, over and above the classical result, of order β and β^2. The exact propagators are then those given in [9, 10] and in our conventions in [22]. Note that employing the vertices of (2.13) and propagators of table 1 along with the above rule for products of distributions even the overall normalisation of the path integral as given in (2.5) is exactly correct.

Our task then is to rewrite the operator $H = -\beta^2\not{D}^2/2$ in Weyl ordered form, i.e. in terms of symmetrised products of the bose operators x^μ and p_μ and anti-symmetrised products for the fermions ψ^r. In particular, for the fermions, this just

[3]Note that the exponent of the path integral is then $-(1/\beta)\int\mathcal{L}$ so that if one continues the Euclidean time $t \to it$ back to real time the exponent becomes $i(1/\beta)\int(p_\mu\dot{x}^\mu + (i/2)\psi_r\dot{\psi}^r - H)$, i.e. none other than the usual Legendre transformation relating lagrangean and hamiltonian.

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means that all products of Dirac matrices should be written in the basis $\gamma^r, \gamma^{rs}, \gamma^{rst}$ and γ^{rstu}. This computation will involve three different spin connections so to avoid confusion let us spell out our notations. The Dirac operator is

$$\not{D} = \not{D}(e, A) = \gamma^\mu D(\omega)_\mu\,, \qquad \gamma^\mu = e_r{}^\mu \gamma^r\,, \qquad D(\omega)_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu rs}\gamma^{rs}\,. \tag{2.9}$$

Here $D(\omega)_\mu$ denotes an operator depending on the spin connection $\omega_{\mu rs} = \omega(e)_{\mu rs} + A_{\mu rs}$ where $\omega(e)_{\mu rs}$ is the torsion free connection and $A_{\mu rs}$ is a completely antisymmetric contortion tensor. The classical (β independent) contribution to H has been computed in [26] and yields the hamiltonian for a spinning particle depending on a third connection $\widehat{\omega}_{\mu rs} = \omega_{\mu rs} + 2A_{\mu rs} = \omega(e)_{\mu rs} + 3A_{\mu rs}$ with contortion three times the usual one. Hence we introduce the operator $D(\widehat{\omega})_\mu$ identical to the previous definition except depending on the new connection $\widehat{\omega}_{\mu rs}$. We reserve the symbol $D_\mu (\equiv e_\mu{}^r D_r)$ for the torsion free covariant derivative.

We now make some simple algebraic manipulations

$$\not{D}^2/2 = \frac{1}{2}\Big(g^{\mu\nu}D(\omega)_\mu - \Gamma^\nu{}_\mu{}^\mu + \gamma^{\mu\nu}D(\omega)_\mu + A_{rs}{}^\nu\gamma^r\gamma^s\Big)D(\omega)_\nu\,, \tag{2.10}$$

$$\begin{aligned} &= \frac{1}{2}\,g^{\mu\nu}\Big(D(\widehat{\omega})_\mu D(\widehat{\omega})_\nu - \Gamma_{\mu\rho}{}^\rho D(\widehat{\omega})_\nu\Big) + \\ &\quad + \frac{1}{16}\Big(R(\omega)_{\mu\nu rs} - 2A^\rho{}_{\mu\nu}A_{\rho rs}\Big)\gamma^{\mu\nu}\gamma^{rs} + \frac{1}{4}(D_\mu A^\mu{}_{rs})\gamma^{rs}\,, \end{aligned} \tag{2.11}$$

with $R_{\mu\nu rs}$ the curvature built from the spin connection $\omega_{\mu rs}$ (our conventions for the Riemann tensor in the presence of torsion are spelled out in the appendix). The remainder of the Weyl ordering computation for the first term in (2.11) follows the computation in [10] and the remaining terms require only the identity $\gamma^{\mu\nu}\gamma_{rs} = \gamma^{\mu\nu}{}_{rs} + 4\,\gamma^{[\mu}{}_{[s}e^{\nu]}{}_{r]} + 2\,e^{[\mu}{}_s e^{\nu]}{}_r$. Hence we find, making the identifications in (2.4)

$$\begin{aligned} H = &\frac{1}{2}\left[g^{\mu\nu}\Big(p_\mu - \frac{i}{2}\widehat{\omega}_{\mu rs}\psi^r\psi^s\Big)\Big(p_\nu - \frac{i}{2}\widehat{\omega}_{\nu tu}\psi^t\psi^u\Big)\right]_W - \frac{1}{2}e^\mu{}_r(D_\mu A_{stu})\Big[\psi^r\psi^s\psi^t\psi^u\Big]_W - \\ &- \frac{\beta^2}{8}\left(\Gamma^\sigma{}_{\mu\rho}\Gamma^{\rho\mu}{}_\sigma + \frac{1}{2}\,\widehat{\omega}_{\mu rs}\widehat{\omega}^{\mu rs} + 3\,A_{rst}A^{rst}\right), \end{aligned} \tag{2.12}$$

where $[\dots]_W$ denotes Weyl ordering. We may now take the Weyl ordered expression for H and replace operators by c-numbers. Integrating out the momenta p_μ one finds the lagrangean

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\,g_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu + \frac{1}{2}\psi^r\left(\delta_{rs}\frac{\mathrm{d}}{\mathrm{d}t} + \dot{x}^\mu\widehat{\omega}_{\mu rs}\right)\psi^s - \frac{1}{2}\,D_r A_{stu}\,\psi^r\psi^s\psi^t\psi^u - \\ &- \frac{\beta^2}{8}\left(\Gamma^\sigma{}_{\mu\rho}\Gamma^{\rho\mu}{}_\sigma + \frac{1}{2}\widehat{\omega}_{\mu rs}\widehat{\omega}^{\mu rs} + 3\,A_{rst}A^{rst}\right). \end{aligned} \tag{2.13}$$

At this point the vertices of the theory are those defined by (2.13) and propagators along with products of distributions are those found according to the midpoint rule.

Observe that the term at order β vanishes identically. For zero torsion, the above result has appeared in [27]. Note also that the first two $O(\beta^2)$ terms in the second line of (2.13) are neither general coordinate (GC) nor local Lorentz (LL) invariant. From first principles however, we know that the result for the index must enjoy these invariances, since varying the background fields $e_\mu{}^r$ and $\omega_{\mu rs}$ with respect to GC and LL transformations, the operator $\not{D}^2$ varies into a commutator $[\not{D}^2, G+L]$, (G and L are the generators of GC and LL transformations) which vanishes under the trace as $[\gamma^5, G+L] = 0 = [\gamma^5, \not{D}^2]$. The resolution of this apparent dichotomy is simply that the propagators, based on the midpoint rule obtained by Weyl ordering, are also not GC or LL covariant. Nonetheless, as we shall see, the combination of these two "evils" will yield a GC and LL invariant result.

2.2 Loop expansion of the path integral

We are now ready to compute the path integral via a loopwise expansion in the Planck constant β to orders β^{-1} and β^0 employing the Feynman rules for the finite-time path integral of [9, 10]. The split into free and interacting parts is made via the expansion around backgrounds y^μ and Ψ so that $x^\mu \to y^\mu + q^\mu$ and $\psi^r \to \Psi^r + \eta^r$. The spin connection $\widehat{\omega}$ is now an independent background field hence in our diagrammatic computation we make a Riemann normal coordinate expansion for the metric only (details can be found in the references just mentioned as well as in [22]). We have listed the propagators in table 1 and some of the required integrals in table 2. Other useful diagrammatic tricks and identities for this type of computation have been developed in [22].

We only need compute graphs quartic in background fermions Ψ^r (in the diagrams they are denoted by a dash on the external line) in order to saturate the Grassmann integration $\mathrm{d}^4\Psi$ in (2.5) which in turn yields a four-dimensional Levi-Civita symbol. There are several "tree level" contributions[4] (those contributions arising from the terms in the action that are independent of quantum fields). To order β^0,

$$\Delta_{\text{tree}}(\not{D}) = \int_{\mathcal{M}} \mathrm{d}^4 y \, \frac{\sqrt{g}}{(2\pi i)^2} \, \epsilon^{rstu} \left[\frac{1}{2\beta} D_r A_{stu} + \frac{1}{32} D_r A_{stu} \, \widehat{\omega}_{\nu mn} \widehat{\omega}^{\nu mn} + \right.$$
$$\left. + \frac{3}{16} A_{mnp} A^{mnp} \, D_r A_{stu} \right], \tag{2.14}$$

beginning with the leading $1/\beta$ contribution to which we draw the reader's attention. This contribution is proportional to the so-called Nieh-Yan tensor [28, 29]. As expressed above, it is clear that this tensor vanishes for exact torsion; alternatively

[4]To avoid confusion, note that we only compute contributions of order β^{-1} and β^0, however, since we must compute the partition function $Z[y^\mu, \Psi^r]$, the generating function of *all* diagrams, disconnected products of "two loop" and "tree" graphs will produce β independent results which must not be neglected.

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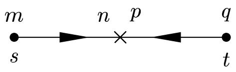
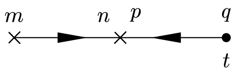
$$= \langle\, q(s)^\mu q(t)^\nu \,\rangle \; = g^{\mu\nu}\Big[t(1-s)\theta(s-t)+s(1-t)\theta(t-s)\Big]\,,$$

$$= \langle\, \eta(s)^m \eta(t)^n \,\rangle = \frac{1}{2}\eta^{mn}\Big[\theta(s-t)-\theta(t-s)\Big]+K^{mn}\,.$$

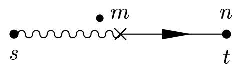
Table 1: Propagators for the action (2.13). The symbol K^{mn} parametrises the choice of complexification of the Majorana spinors, as explained in [10].



$$= \delta^{mn}\delta^{pq}\big[\frac{1}{2}(s-t)\,\epsilon(s-t)-\frac{1}{4}\big]+ \\ +(s-\frac{1}{2})\,\delta^{mn}K^{qp}+(t-\frac{1}{2})\,\delta^{qp}K^{mn}+K^{mn}K^{qp}\,,$$

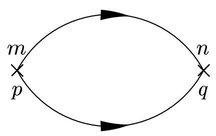


$$= \frac{1}{2}(t^2-t)\delta^{mn}\delta^{pq}-(t-\frac{1}{2})\delta^{pq}K^{mn}+K^{mn}K^{qp}\,,$$

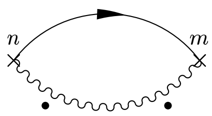


$$= \; -$$

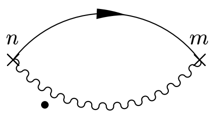
.



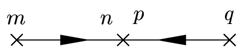
$$= \frac{1}{4}\delta^{mn}\delta^{pq}+K^{mn}K^{pq}\,,$$



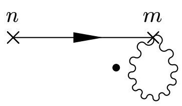
$$= 0\,,$$



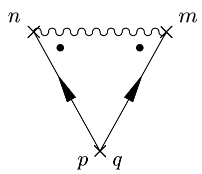
$$= -\frac{1}{6}\delta^{nm}\,,$$



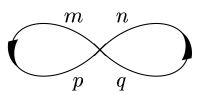
$$= -\frac{1}{12}\delta^{mn}\delta^{pq}+ \\ +K^{mn}K^{qp}\,,$$



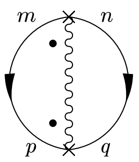
$$= \delta^{nm}\frac{1}{12}\,,$$



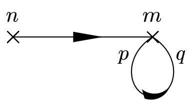
$$= -\frac{1}{6}\delta^{nm}\delta^{pq}\,,$$



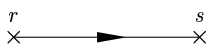
$$= K^{mp}K^{qn}\,,$$



$$= -\frac{1}{4}\delta^{mp}\delta^{nq}\,,$$



$$= K^{nm}K^{pq}\,.$$



$$= 0\,,$$

Table 2: A number of useful integrals obtained by straightforward application of the propagators in table 1 where the definition of K^{mn} may also be found. The remaining integrals required for our computation as well as a detailed explanation of the diagrammatic notation above can be found in [22].

one can re-express the Nieh-Yan contribution into a perhaps more familiar form by writing

$$2\,\partial_{[\mu}A_{\nu\rho\sigma]} = R(\omega)_{[\mu\nu\rho\sigma]} - 2\,A_{\lambda[\mu\nu}A^{\lambda}{}_{\rho\sigma]}\,. \tag{2.15}$$

where $R(\omega)_{\mu\nu\rho\sigma}$ is the Riemann curvature of the torsion-full connection $\omega_{\mu rs} = \omega(e)_{\mu rs} + A_{\mu rs}$. We will discuss this contribution, which has led to some confusion in the literature, in more detail in the conclusion. Observe that the order β^0 terms are also proportional to the curl of the torsion, though not immediately recognisable as a topological invariant as they are only a part of the result at this order in the Planck constant. Furthermore, observe that the product of the Nieh-Yan tree level tree level β^{-1} contribution and the Christ and Lee Weyl ordering counter terms yield the second and third $\mathcal{O}(\beta^0)$ terms in (2.14). This provides the long sought after explicit confirmation that these two loop counter terms (they appear with a factor β^2 in the action (2.13)) are actually necessary in a rigorous definition of the path integral.

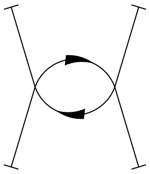
The remaining graphs involve one- and two-loop integrals and are given in tables 3 and 4. Note that we have not drawn graphs obtained by different locations of the dots, which denote time derivatives on propagators, although these contributions are, of course, included in the quoted results. Adding all these contributions together our result for the regulated trace of γ_5 is

$$\begin{aligned}\Delta(\not{D}\,) = \int_{\mathcal{M}} \mathrm{d}^4y\,\frac{\sqrt{g}}{(2\pi i)^2}\,\epsilon^{rstu}\Big\{\frac{1}{2\beta}\,D_rA_{stu} + \frac{1}{192}\Big[R(\widehat{\omega})_{\mu\nu rs}R(\widehat{\omega})^{\mu\nu}{}_{tu} + \\ + 8\,D^{\mu}D_{\mu}D_rA_{stu} + 36\,A_{mnp}A^{mnp}\,D_rA_{stu} + 4\,R\,D_rA_{stu}\Big]\Big\} + \mathcal{O}(\beta)\,.\end{aligned} \tag{2.16}$$

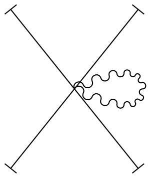
Note that the contracted indices in the term quadratic in the Riemann tensors are *curved* ones since in the presence of torsion the $R(\widehat{\omega})_{\mu\nu rs} \neq R(\widehat{\omega})_{rs\mu\nu}$ (see (A.15)). The graphs for this term are not displayed in the table but can be easily computed along the lines of [22]. Also, observe that the scalar curvature on the second line of (2.16) is that of the torsion-free Riemann curvature since it comes from the figure eight graph on the last line of table 4 in which one expands the metric $g_{\mu\nu}(y+q)$ to second order in quantum fluctuations $q(t)^{\mu}$ which in Riemann normal coordinates yields the torsion free Riemann curvature.

A useful test of the result (2.16) is independence of the choice of complexification of the original Majorana spinors required to employ a coherent state formalism for the fermions. The three terms from the one-loop diagrams and the single term from the two-loop diagram do indeed vanish upon using the Schouten identity, by virtue of the fact that K^{mn} is anti-symmetric.

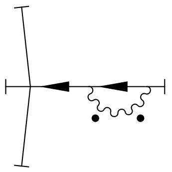
To facilitate comparison with existing results and see that the result is a total derivative it is useful to extract the dependence on the torsion free curvature which



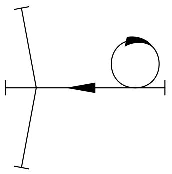
$$= -\frac{1}{64}\,\Psi^{rstu}\,\widehat{F}_{rsnm}\widehat{F}_{tu}{}^{mn} + \frac{1}{16}\Psi^{rstu}\widehat{F}_{rsnm}\widehat{F}_{tupq}K^{mp}K^{nq}\,,$$



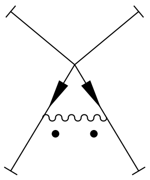
$$= -\frac{1}{288}\,\Psi^{rstu}\,\partial^{\mu}\partial_{\mu}\widehat{F}_{rstu}\,,$$



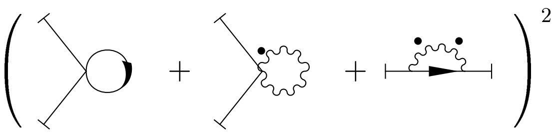
$$= \frac{2}{288}\Psi^{rstu}\widehat{F}_{rstn}\widehat{\omega}^{\mu nm}\widehat{\omega}_{\mu um}\,,$$



$$= -\frac{24}{288}\Psi^{rstu}\widehat{F}_{rstn}\widehat{F}_{upqm}K^{np}K^{qm}\,,$$



$$= -\frac{12}{288}\Psi^{rstu}\widehat{F}_{rs}{}^{nm}\widehat{\omega}_{\mu tm}\widehat{\omega}^{\mu}{}_{un}\,,$$

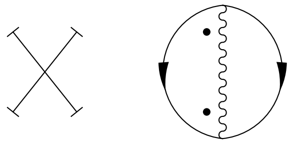


$$= \frac{1}{32}\Psi^{rstu}\widehat{F}_{rsmn}K^{mn}\widehat{F}_{tupq}K^{pq}\,,$$

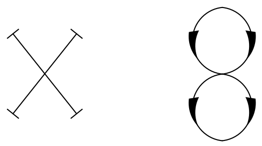
Table 3: One-loop diagrams contributing to the path integral at order β^0. Only those graphs that are not already zero because of symmetry properties of the background fields are displayed. Note that we denote $\Psi^{rstu} \equiv \Psi^r\Psi^s\Psi^t\Psi^u$.

we denote by $R_{\mu\nu\rho}{}^{\sigma}$ where $R_{\mu\nu\rho}{}^{\sigma}V_{\sigma} = [D_{\mu}, D_{\nu}]V_{\rho}$. Furthermore, we rewrite the result in terms of the axial contortion vector $A_r = \epsilon_{rstu}A^{stu}$. Using (A.15) one derives that

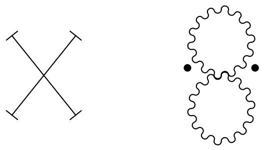
$$\begin{aligned} R(\widehat{\omega})_{\mu\nu rs}R(\widehat{\omega})^{\mu\nu}{}_{tu}\,\epsilon^{rstu} &= \Big[R_{\mu\nu\rho\sigma}R_{\kappa\lambda}{}^{\rho\sigma} + 2\,D_{\mu}A_{\nu}D_{\kappa}A_{\lambda}\Big]\,\frac{1}{\sqrt{g}}\,\epsilon^{\mu\nu\kappa\lambda} - \\ &\quad -4\Big[D^{\mu}A_{\mu}A^{\nu}A_{\nu} - D_{\mu}A_{\nu}A^{\mu}A^{\nu}\Big] - 8\,D_{\mu}A_{\nu}R^{\mu\nu}\,. \quad (2.17) \end{aligned}$$



$$= -\frac{(3/4)}{288}\widehat{\omega}_{\mu mn}\widehat{\omega}^{\mu mn}\Psi^{rstu}\widehat{F}_{rstu}\,,$$



$$= \frac{1}{192}\widehat{F}_{mnpq}K^{mn}K^{pq}\Psi^{rstu}\widehat{F}_{rstu}\,,$$



$$= 4\,R\,\widehat{F}_{rstu}\Psi^{rstu}\,.$$

Table 4: Disconnected two-loop diagrams contributing to the path integral at order β^0. Note that the result for the diagram on the third line includes graphs obtained by permutation of the time derivatives. One must also add to this diagram the corresponding ghost diagrams which cancel the divergence ${}^\bullet\!\Delta^\bullet(s,s) = \frac{\mathrm{d}^2}{\mathrm{d}s\mathrm{d}t}\Delta(s,t)\Big|_{s=t} = \delta(0) - 1$.

The cross terms between R and the torsion vanish identically by virtue of the Ricci symmetry $R_{\mu[\nu\rho\sigma]} = 0$ and so do the terms quartic in the torsion. The second line of (2.16) is easily written as a covariant derivative by commuting the triple derivatives using

$$D^\mu D_\mu D_\nu A^\nu = D_\nu D^\mu D_\mu A^\nu + D_\mu A_\nu R^{\mu\nu} + \frac{1}{2}A^\mu D_\mu R\,. \tag{2.18}$$

This brings (2.16) into a form derived previously by Obukhov [7] using heat kernel methods,[5]

$$\Delta(\not{D}\,) = \int_{\mathcal{M}} \mathrm{d}^4y\,\frac{\sqrt{g}}{(2\pi i)^2}\left\{\frac{1}{2\beta}D_\mu A^\mu + \frac{1}{192}\frac{1}{\sqrt{g}}\,\epsilon^{\rho\sigma\kappa\lambda}\left[R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\kappa\lambda} + \frac{1}{2}F_{\rho\sigma}F_{\kappa\lambda}\right] + \right.$$
$$\left. + \frac{1}{24}\,D_\mu\mathcal{K}^\mu\right\} + \mathcal{O}(\beta)\,, \tag{2.19}$$

with

$$\mathcal{K}^\mu = \left(D^\nu D_\nu + \frac{1}{4}A^\nu A_\nu + \frac{1}{2}R\right)A^\mu\,, \qquad F_{\mu\nu} = 2D_{[\mu}A_{\nu]}\,. \tag{2.20}$$

[5] Obukhov's work has been re-examined by Yajima and Kimura [30, 31] who found, again in a heat kernel approach, the same result for the bulk index with torsion but disagree upon the precise details of his derivation of the heat kernel coefficients and claim that it leads to incorrect results for the Lorentz anomaly in the presence of torsion. We do not wish to enter this debate here and it suffices to say that the path integral techniques presented in this paper can equally well be employed for other anomalies and could, therefore, provide an independent resolution of this issue.

In the case that the Nieh-Yan tensor vanishes, which is manifestly so when the torsion is exact, the above expression (2.16) reduces to the term quadratic in $R(\widehat{\omega})$ which coincides with the result derived by Mavromatos some time ago [26].

As long as the $\mathcal{K}^\mu$ vector is smooth (as it is for smooth metric and torsion), the additional contribution $D_\mu \mathcal{K}^\mu$ to the bulk index in the presence of torsion will only receive contributions from the boundary. The subtle interplay between the three order β^0 terms in (2.19) can be clearly seen in the explicit example given in subsection 4.4.1 below.

We stress that the Nieh-Yan term in the action (2.13) along with the Christ and Lee two-loop Weyl ordering counter terms have been crucial in obtaining this result (see the two-loop diagrams in table 4 as well as (2.14)). Although the $1/\beta$ Nieh-Yan bulk contribution to index has appeared in the literature before [15, 16, 17, 18, 19, 20], in our path integral approach one sees clearly its importance for the order β^0 terms and hence the necessity to include it when computing the bulk index in the presence of torsion. We will comment further on this term in the conclusion.

Finally, we observe that the folkloric statement; "the chiral anomaly is topological and therefore insensitive to the subtleties of the precise definition of the path integral" can be made very definitive in the approach employed above. The reason being that in the absence of torsion the different counter terms corresponding to varying ordering schemes have no affect on the β^0 result since they are all higher loop effects. Only once the torsion is turned on, so that the lagrangean includes the $1/\beta$ Nieh-Yan contribution must one worry about the subtleties of ordering schemes etc. when computing the β independent anomaly.

3. Boundary contributions to the index with torsion

3.1 The APS boundary correction with torsion

On manifolds with boundary, the central question is which boundary conditions one must impose for spinors in order that the index problem is well posed. Exactly this problem was solved by Atiyah, Patodi and Singer in the mid-seventies [1] for manifolds which have a product form near the boundary (i.e. when the manifold is a cylinder with the boundary manifold as cross section). In this case the Hilbert space is sufficiently under control and consistent boundary conditions can be imposed on spinors. Moreover, due to the fact that the radial factor can be written down explicitly, the index computation can be reduced to the problem of suitably counting the difference between the number of positive and negative eigenvalues of the boundary Dirac operator.

For manifolds that are not of a product form, a split of the Hilbert space is not as obvious. Nevertheless, due to the topological nature of the index, it is possible to deform these manifolds until their boundary takes a product form. The index then splits into a bulk part plus the APS boundary term, amended by a modification due to the deformation. This modification was pointed out by Gilkey [2] and we will generalise it here to also include torsion.

Let us now explain these matters in more detail. Suppose $\partial = \partial/\partial r$ is a vector normal to the boundary $\partial\mathcal{M}$ of the manifold in question. Then the Dirac operator (possibly including a coupling to the antisymmetric part of the contortion, assumed to have only components parallel to the boundary) takes the form

$$\not{D}(e,A) = \gamma^{[r]}\left[\left(\partial + \frac{1}{2}\omega(e)^{\rho}{}_{\rho[r]}\right) + \mathcal{B}\right], \tag{3.1}$$

where $\mathcal{B}$ evaluated at the boundary $r = r_b$ is the boundary Dirac operator. We denote the curved index in the normal direction spanned by the coordinate r as $[r]$ and $\omega(e)^{\rho}{}_{\rho[r]}$ is the trace of the torsion-free part of the spin connection computed from the vierbein $e_{\mu}{}^{r}$ (no confusion should arise between the flat index r and the radial coordinate r). Observe that since $\not{D}(e,A)$ is antihermitean with respect to the volume element $\sqrt{g(e)}$ the coefficient of the single outward pointing Dirac matrix $\gamma^{[r]} = \gamma^{r} e_{r}{}^{[r]}$ is fixed to be that appearing in (3.1). However, to verify that the Dirac operator is antihermitean with respect to the inner product $(\Xi, \Psi) = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{g(e)}\; \Xi^{\dagger}\,\Psi$, along with taking into account differentiations by parts that hit the measure factor $\sqrt{g(e)}$ one must also ensure that any surface terms also vanish by imposing appropriate boundary conditions on spinors. However, local Neumann or Dirichlet boundary conditions do not lead to solutions of the first order homogeneous zero-mode Dirac equation. Instead non-local boundary conditions are required.

Let us write the Dirac operator (3.1) in a chiral basis for the Dirac matrices

$$\not{D} = \frac{1}{V}\begin{pmatrix} 0 & \left(\partial + \frac{1}{2}\omega(e)^{\rho}{}_{\rho[r]}\right) + B \\ \left(\partial + \frac{1}{2}\omega(e)^{\rho}{}_{\rho[r]}\right) - B & 0 \end{pmatrix}, \tag{3.2}$$

where V is some function of the normal variable r. The 2×2 matrix B is hermitean acting on spinors on the compact boundary manifold $\partial\mathcal{M}$. The APS split of the index computation into a bulk piece and a boundary piece can only be achieved if (3.2) takes a simpler form near the boundary. First of all, B has to be independent of r and the connection trace $\omega(e)^{\rho}{}_{\rho[r]}$ should vanish. This holds when the manifold is a product metric near the boundary. In addition we will require that there is no "radial" component of the torsion (i.e. $A_{[r]\nu\rho} = 0$) as this would enter the Dirac operator in a way similar to the connection trace and prohibit an explicit solution of the radial part of the eigenfunction. Although the APS condition can be imposed by smoothly deforming the manifold near the boundary, it is not obvious that such a deformation argument can be used to set a radial torsion component to zero.

When these conditions are satisfied, Atiyah et al. found that one can consistently impose the following non-local boundary conditions on spinors Ψ,

$$\left.\begin{array}{l}\displaystyle\sum_{l\geq 0}|\,l\,\rangle\langle\,l|\,\phi(r_b)=0\\ \displaystyle\sum_{l<0}|\,l\,\rangle\langle\,l|\,\chi(r_b)=0\end{array}\right\}\qquad \Psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \tag{3.3}$$

where the boundary is at $r=r_b$ and the two component spinors $|\,l\,\rangle$ are eigenspinors of B,

$$B\,|l\,\rangle = l\,|l\,\rangle\,. \tag{3.4}$$

Only for such non-local boundary conditions does a non-trivial set of zero-modes of the Dirac operator survive. The index splits in a bulk and a boundary piece as indicated according to

$$\mathrm{index}\not{D} = \mathrm{index(bulk)}\ -\ \frac{1}{2}\Big(\eta_B(0)+h_B\Big)\,, \tag{3.5}$$

where index(bulk) is the usual integral over the second Pontrjagin class, h is the number of harmonic spinors on the boundary manifold $\partial\mathcal{M}$, i.e. the number of solutions to $B\,\Psi=0$, and $\eta_B(s)$ is a spectral invariant of the boundary operator B given by

$$\eta_B(s)=\sum_{l\neq 0}\frac{\mathrm{sign}(l)}{|l|^s} \tag{3.6}$$

where the sum runs over the eigenvalues l of the boundary Dirac operator B. As a function of the complex variable s, $\eta_B(s)$ has a well defined analytic continuation to $s=0$ [1, 32, 33].

When B only has a finite number of eigenvalues one can take $s=0$ before summing (3.6) (for other cases in which the APS conditions can be made explicit see e.g. [34, 35]). In this case, one can understand the appearance of $\eta_B(0)$ in the expression for the index rather easily. Assuming that near the boundary at $r=r_b$ the manifold is a product metric, the zero mode equation on this cylinder can be written as

$$\begin{pmatrix}0 & \partial+B\\ \partial-B & 0\end{pmatrix}\begin{pmatrix}c_k\,e^{kr}|l\rangle\\ d_k\,e^{kr}|l\rangle\end{pmatrix}=\begin{pmatrix}(k+l)\,d_k\,e^{kr}|l\rangle\\ (k-l)\,c_k\,e^{kr}|l\rangle\end{pmatrix}=0 \tag{3.7}$$

(for the product metric we may simultaneously diagonalise B and ∂). This implies that positive chirality zero-modes (those for which only the upper components are non-vanishing) have $k=l$ while those of negative chirality have $k=-l$. The boundary conditions (3.3) imply that the positive chirality states have only negative eigenvalues of B while the negative chirality states carry positive B-eigenvalues (due to the relation between k and l this implies that only zero-modes which vanish exponentially as $r \to \infty$ remain). The contribution to the index from the cylinder is therefore the disparity between the number of negative and positive eigenvalues of B which is clearly given by (3.6) at $s = 0$ and this limit may be taken before performing the finite summation (the factor $1/2$ in (3.5) arises from the fact that half of the spinor boundary modes do not continue smoothly into the interior [1, 34]). The APS theorem renders this argument rigorous for operators B with an infinite number of eigenvalues and also shows that the index problem indeed splits in a bulk and boundary piece according to (3.5) by appropriately gluing the two pieces of the manifold.

Before we discuss the Gilkey correction term, we will now first show how the η invariant can be computed for a class of boundary Dirac operators general enough to include the torsion-full examples to be discussed later.

3.2 Computation of the η invariant for manifolds with torsion and S^3 boundary

The η invariant must be computed on a case by case basis and since it involves all non-zero boundary eigenvalues this can be rather tricky. Fortunately this computation has been carried out by Hitchin for a large class of boundary geometries [36] (to be specific, squashed S^3 metrics). The addition of torsion modifies the boundary Dirac operator by what amounts to only a slight generalisation of Hitchin's calculations. However, in keeping with the rest of this paper and because the explicit solutions we find to the boundary Dirac equation will be needed when we explain how one is able to compare index problems for manifolds with mutually anticommuting Dirac operators in section 4, we will now present these additional details.

We begin with a boundary operator of the form

$$\mathcal{B} = S\left(\gamma^{34}\Sigma_1 + \gamma^{42}\Sigma_2\right) + \frac{1}{\lambda}\gamma^{14}\Sigma_3 - \gamma^{1234}\left[\frac{\lambda^2+2}{4\lambda} + Z\right]. \tag{3.8}$$

where λ, S and Z are arbitrary parameters (depending, in general, on r_b). The operators Σ_i commute with the SO(3) Killing vectors (i.e. they are the inverses of the left-invariant forms) and satisfy the algebra

$$[\Sigma_1, \Sigma_2] = \Sigma_3 \quad \text{and cyclic.} \tag{3.9}$$

To motivate the somewhat obscure looking form we have chosen for the operator (3.8), we note that it is exactly the case solved by Hitchin [36] when $S = Z = 0$. The generalised form presented here can be handled too and (as we will see) appears in the example manifolds discussed in section 4. A useful basis for the Dirac matrices is

$$\gamma^1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i\sigma_x \\ i\sigma_x & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{pmatrix}, \tag{3.10}$$

One then has

$$\mathcal{B} = \begin{pmatrix} P(S,Z) & 0 \\ 0 & -P(-S,Z) \end{pmatrix}, \tag{3.11}$$

where the "Hitchin-operator" is given by

$$P(S,Z) = iS[\Sigma_1\sigma_x + \Sigma_2\sigma_y] + \frac{i}{\lambda}\Sigma_3\sigma_z + \frac{\lambda^2+2}{4\lambda} + Z\,. \tag{3.12}$$

Therefore it is most useful to solve the eigenvalue problem $P(S,Z)\,\varphi = z\varphi$. To begin with call $\Sigma_i = -iJ_i$ where J_i is the angular momentum operator ($i = 1,\ldots,3$) (and we employ the usual eigenstates $|j,m\rangle$ of J^2 and J_3). Note that the Hitchin operator commutes with the Casimir $[J^2, P(S,Z)] = 0$ so we may look for solutions at a fixed j. Actually, it is easy to solve this equation with the following ansatz

$$\varphi = \begin{pmatrix} \alpha\,|j,m\rangle \\ |j,m+1\rangle \end{pmatrix}, \tag{3.13}$$

where we define $|j,j+1\rangle = 0 = |j,-j-1\rangle$. Then if $m = j$ or $m = -j-1$ we clearly obtain solutions with eigenvalue

$$z = j/\lambda + (\lambda^2+2)/4\lambda + Z\,. \tag{3.14}$$

Now if $j > m > -j-1$ so that $\Delta \equiv \sqrt{(j-m)(j+m+1)}$ is real and non-vanishing then we have solutions with eigenvalues

$$z_\pm = \alpha_\pm\,S\Delta - (m+1)/\lambda + (\lambda^2+2)/4\lambda + Z\,, \tag{3.15}$$

where $\alpha_\pm$ solves the equation

$$\alpha_\pm^2 - \frac{2m+1}{\Delta S\lambda}\,\alpha_\pm - 1 = 0\,, \tag{3.16}$$

so that explicitly

$$z_\pm = \frac{\lambda^2 \pm 2\sqrt{(2m+1)^2 + 4S^2\lambda^2(j-m)(j+m+1)}}{4\lambda} + Z\,. \tag{3.17}$$

To conform with the notation of Hitchin we relabel $j = (p+q-1)/2$ and $m = (p-q-1)/2$ so that the eigenvectors, eigenvalues and their multiplicities read as in table 5, where the integers p and q take all values strictly greater than zero

eigenvector	*eigenvalue*	*multiplicity*
$\begin{pmatrix} \lvert\frac{p-1}{2},\frac{p-1}{2}\rangle \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \lvert\frac{p-1}{2},\frac{1-p}{2}\rangle \end{pmatrix}$	$\frac{p}{2\lambda}+\frac{\lambda}{4}+Z$	$2p$
$\begin{pmatrix} \frac{p-q\pm\sqrt{4pqS^2\lambda^2+(p-q)^2}}{2\lambda\sqrt{pq}}\lvert\frac{p+q-1}{2},\frac{p-q-1}{2}\rangle \\ \lvert\frac{p+q-1}{2},\frac{p-q+1}{2}\rangle \end{pmatrix}$	$\frac{\lambda}{4}\pm\frac{\sqrt{4pqS^2\lambda^2+(p-q)^2}}{2\lambda}+Z$	$p+q$

Table 5: Eigenvectors, eigenvalues and multiplicities.

(note that the multiplicities above refer to distinct eigenvalues). To understand the multiplicities quoted in table 5, it is useful to briefly explain the representation theory of states on S^3. The metric on the squashed S^3 (considered in this section) may be represented (up to an overall irrelevant scaling) in terms of left invariant forms Σ^i ($i = 1, \ldots, 3$) as

$$ds^2\big|_{S^3} = (\Sigma^1)^2 + (\Sigma^2)^2 + (\Sigma^3/(S\lambda))^2\,. \tag{3.18}$$

From the invariant forms one may write down left invariant vectors Σ_i which commute with the Killing vectors k_i and satisfy the algebra (the explicit forms of these vectors are given in section 4.3)

$$\begin{aligned} [\Sigma_1, \Sigma_2] &= \Sigma_3 \quad \text{and cyclic,} \\ [\Sigma_i\,, k_j] &= 0\,, \\ [k_1\,, k_2] &= k_3 \quad \text{and cyclic.} \end{aligned} \tag{3.19}$$

Representing the operators Σ_i on the space $\{|j,m\rangle\}$ as above and the Killing vectors on a second copy of this space $\{|J,M\rangle\}$ then from the relation

$$\sum_{i=1}^{3}(\Sigma_i)^2 = \sum_{i=1}^{3}(k_i)^2 \tag{3.20}$$

expressing equality of the Casimirs on S^3 (which may be easily verified from the explicit forms in 4.3), one finds the representation space of functions on S^3 to be $\{|j,m\rangle \otimes |j,M\rangle\}$ which yields the multiplicities quoted above (since in table 5 we suppressed the dependence on the "right representation" $|J,M\rangle$).

We are now left with the task of computing $\eta_B(0) = \eta_P(0)$ as in (3.6) given the spectrum in table 5. Therefore we must find the analytic continuation to $s = 0$ of (we drop an overall factor $(2\lambda)^s$ which clearly does not modify $\eta_B(0)$)

$$\eta_B(s) = \sum_{p>0} \frac{2p}{(p+\lambda^2/2+2Z\lambda)^s} + \sum_{p,q>0} \left\{ \frac{p+q}{\left(\lambda^2/2+\sqrt{D(\lambda S,p,q)}+2Z\lambda\right)^s} - \right. \tag{3.21}$$

$$\left. - \frac{p+q}{\left(-\lambda^2/2+\sqrt{D(\lambda S,p,q)}-2Z\lambda\right)^s} \right\},$$

where $D(\lambda S, p, q) = 4pq(\lambda S)^2 + (p-q)^2$. (Note that in all cases that we are interested, λ, S and Z are such that $-\lambda^2/2 + \sqrt{D(\lambda S,p,q)} - 2Z\lambda > 0$, which motivates the above split in a sum over positive and negative eigenvalues, but in general one should of course worry about this separation as well as any zero-modes of the boundary operator). The single sum can be expressed in zeta functions without any difficulty. The double sum requires a lengthy analysis, most easily performed by first Taylor expanding in powers of the new variable Λ defined by $\Lambda^2/2 = \lambda^2/2 + 2Z\lambda$,

$$\eta_B(s) = 2\,\zeta(s-1,\Lambda^2/2) - \Lambda^2\,\zeta(s,\Lambda^2/2) - \tag{3.22}$$

$$-2s\left(\frac{\Lambda^2}{2}\right) f\big((s+1)/2\big) - \frac{2s(s+1)(s+2)}{3!}\left(\frac{\Lambda^2}{2}\right)^3 f\big((s+3)/2\big) + \cdots$$

where the dots vanish for $s=0$ and the function $f(s)$ is defined by

$$f(s) = \sum_{p,q>0} \frac{p+q}{D(\lambda S,p,q)^s}\,. \tag{3.23}$$

The terms involving the zeta functions yield $\Lambda^4/4 - 1/6$ while the analysis of the residues of $f(s)$ is rather nontrivial and produces [37]

$$\begin{aligned} \mathrm{res}\left[f\left(\frac{s+1}{2}\right)\right]_{s=0} &= \frac{1}{3}\left((\lambda S)^2 - 1\right), \\ \mathrm{res}\left[f\left(\frac{s+3}{2}\right)\right]_{s=0} &= \frac{1}{(\lambda S)^2}\,. \end{aligned} \tag{3.24}$$

This finally yields the η invariant at $s=0$

$$\begin{aligned} \eta_B(0) = &-\frac{1}{6} - \frac{\lambda}{3}\left[\frac{16\,Z^3}{S^2} - 4\,Z\right] + \frac{\lambda^2}{3}\left[12\,Z^2 - \frac{12\,Z^2}{S^2} + 1\right] - \\ &-\frac{\lambda^3}{3}\left[\frac{3\,Z}{S^2} + 4\,S^2 Z - 6\,Z\right] + \frac{\lambda^4}{4}\left[1 - \frac{4\,S^2}{3} - \frac{1}{3\,S^2}\right]. \end{aligned} \tag{3.25}$$

At $S=1$ and $Z=0$, we recover Hitchin's result $\eta_B(0) = -1/6 + \lambda^2/3 - \lambda^4/6$. So summarising, our final result for the η invariant for the class of Dirac operators in (3.8) is given by the above result (3.25).

3.3 Boundary correction for non-product metrics

The APS analysis presented in the previous two subsections is valid for the manifolds whose metric takes a product form [1] at the boundary (more intuitively, those manifolds which, near the boundary, are a cylinder with the boundary manifold as cross section). If this is not the case, the manifold has to be deformed to such a product structure near the boundary before the above machinery can be applied. The bulk term should in turn also be computed for this *deformed* metric. As an explicit form of the deformed metric is often not available, it is, however, much easier to determine instead the error that one has made by computing the bulk contribution using the original metric. This error term, due to Gilkey [2] for torsion-less manifolds, has to be subtracted from the bulk.

The form of this correction is rather simple. In the absence of torsion, the bulk term can be written as a boundary integral

$$\frac{1}{24.8\pi^2}\int_{\mathcal{M}} R\wedge R = \frac{1}{24.8\pi^2}\int_{\mathcal{M}} \mathrm{d}C = \frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}} C\,, \tag{3.26}$$

provided discontinuities of C are taken into account properly. For the metric deformed to a product structure near the boundary, we get a boundary integral with C replaced by the Chern-Simons term of the product metric $g^\sharp_{\mu\nu} = g_{\mu\nu}(r = r_b)$, $C^\sharp$. Therefore, in the absence of torsion, the correction term that has to be added is

$$\frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}} [C^\sharp - C]\,, \tag{3.27}$$

where the Chern-Simons three-form $C = \omega\wedge R - \frac{1}{3}\omega\wedge\omega\wedge\omega$ (and similarly for $C^\sharp$ built from "sharped" objects computed from the product metric). Note that the integrand of (3.27) can be written as $\theta\wedge R$ where $\theta = \omega - \omega^\sharp$ is the second fundamental form and ω and $\omega^\sharp$ are the torsion-free spin connections computed from the metric and product metrics respectively. (Where we *warn* the reader that from here on we replace $\omega(e)$ by simply ω to denote the torsion-free spin connection.) However, for intuitive, along with practical reasons, we prefer the form given in (3.27).

For the torsion-full case the integrand of the bulk index derived in section 2 above may be written as the exterior derivative of a generalised Chern-Simons form

$$\omega\wedge R - \frac{1}{3}\omega\wedge\omega\wedge\omega - \frac{1}{2}A\wedge F - 2\,\mathcal{K} \equiv C(A) - 2\,\mathcal{K}\,, \tag{3.28}$$

where the three-form $\mathcal{K}$ is the Hodge dual of the vector $\mathcal{K}^\mu$ in (2.20),

$$\mathcal{K}^\mu = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\rho\sigma}\mathcal{K}_{\nu\rho\sigma}\,. \tag{3.29}$$

Therefore, the boundary correction for the torsion-full case is simply

$$\frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}} [C^\sharp(A) - 2\,\mathcal{K}^\sharp - C(A) + 2\,\mathcal{K}]\,, \tag{3.30}$$

where $\mathcal{K}^\mu$ is computed by inserting $r = r_b$ in the axial contortion vector A^μ before computing the covariant derivatives in (2.20).

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3.4 Generalised APS index theorem

Orchestrating the above bulk results presented in (2.19) as well as the η term and the boundary contribution (3.30) we obtain an APS index theorem generalised to manifolds with torsion

$$\begin{aligned} \mathrm{index}\not{D}\,(e,A) = & \frac{1}{24.8\pi^2}\int_{\mathcal{M}}\Big[R(e)^{mn}\wedge R(e)_{nm} - \frac{1}{2}F(A)\wedge F(A) - 2\sqrt{g}\,D_\mu\mathcal{K}^\mu\Big] + \\ & + \frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}}\Big[C^\sharp(A) - 2\,\mathcal{K}^\sharp - C(A) + 2\,\mathcal{K}\Big] - \\ & - \frac{1}{2}\big[\eta_B(0) + h_B\big]\,, \end{aligned} \tag{3.31}$$

with the three-form $\mathcal{K}$ defined by (3.29) and (2.20) above. A result similar to (3.31) has appeared in a string theoretical context in [38] for which one encounters exact torsion and most of the terms above vanish. Needless to say, in the general case presented here there are many extra subtleties as discussed in the preceeding text.

4. Index theorems for manifolds with mutually anticommuting Dirac operators

4.1 General formulation

The existence of mutually anticommuting Dirac operators on a pair of manifolds can be employed to derive a relation between the index theorems on these manifolds which we shall now present in detail. At the end of this section we test this relation on the explicit example of the Taub-NUT and dual Taub-NUT manifolds.

To begin with, given a manifold $\mathcal{M}$ with (possibly torsion-full) Dirac operator $\not{D}\,(e,A)$ satisfying

$$\not{D}\,(e,A) = -\not{D}\,(e,A)^\dagger\,, \qquad \{\not{D}\,(e,A),\gamma^5\} = 0\,, \tag{4.1}$$

then let us assume that we have found a second operator $\not{D}\,(\tilde{e},\widetilde{V})$ acting on spinors defined on $\mathcal{M}$ which solves the equations

$$\{\not{D}\,(e,A),\not{D}\,(\tilde{e},\widetilde{V})\} = 0\,, \tag{4.2}$$

$$\not{D}\,(\tilde{e},\widetilde{V}) = -\not{D}\,(\tilde{e},\widetilde{V})^\dagger\,, \quad \{\not{D}\,(\tilde{e},\widetilde{V}),\gamma^5\} = 0\,. \tag{4.3}$$

At this point we do *not* yet identify $\not{D}\,(\tilde{e},\widetilde{V})$ with the Dirac operator on a manifold $\widetilde{\mathcal{M}}$. Importantly the hermiticity requirement in (4.3) is formulated on the manifold $\mathcal{M}$ (the precise definition of the adjoint operation for spinors on manifolds with boundary was presented in section 3.1). In general, given that the inverse vierbeine $e_r{}^\mu \neq \tilde{e}_r{}^\mu$, observe that it will certainly be necessary to include the additional vector coupling $\widetilde{V}^\mu$ in $\not{D}\,(\tilde{e},\widetilde{V})$ to satisfy both (4.2) and (4.3).

As the operators $-i\not{D}(\tilde{e},\widetilde{V})$ and $-i\not{D}(e,A)$ anticommute, we can construct a new operator $\gamma^5\not{D}(e,A)$ commuting with the original Dirac operator,

$$[-i\not{D}(\tilde{e},\widetilde{V}),\gamma^5\not{D}(e,A)]=0\,. \tag{4.4}$$

These two operators (hermitean on $\mathcal{M}$) can thus be diagonalised simultaneously. Eigenspinors with non-zero eigenvalues occur in pairs for both operators, since they anticommute with γ^5. However, eigenspinors with vanishing $-i\not{D}(\tilde{e},\widetilde{V})$ eigenvalues are not necessarily those with zero $\gamma^5\not{D}(e,A)$ eigenvalues. Nonetheless, in the computation of the index,

$$\text{index}\,\not{D}(\tilde{e},\widetilde{V})=n_+(\not{D}(\tilde{e},\widetilde{V}))-n_-(\not{D}(\tilde{e},\widetilde{V}))\,, \tag{4.5}$$

they still occur in pairs (n_+ denotes the number of zero-modes with positive chirality). The above expression therefore only receives contributions from those zero-modes that are also zero-modes of $\gamma^5\not{D}(e,A)$, that is $\text{Ker}'\,(-i\not{D}(\tilde{e},\widetilde{V}))\subset\text{Ker}'\,(\gamma^5\not{D}(e,A))$, where Ker' denotes the set of zero-modes which do not have a partner of opposite chirality. The same holds true when we interchange the rôle of the two Dirac operators and we therefore also have $\text{Ker}'\,(-i\not{D}(\tilde{e},\widetilde{V}))\supset\text{Ker}'\,(\gamma^5\not{D}(e,A))$. We may therefore conclude that the indices of $-i\not{D}(\tilde{e},\widetilde{V})$ and $\gamma^5\not{D}(e,A)$ are equal and hence[6]

$$\text{index}\,\not{D}(\tilde{e},\widetilde{V})=\text{index}\,\not{D}(e,A)\,, \tag{4.6}$$

since if Ψ is an eigenspinor of the operator $\gamma^5\,\not{D}(e,A)$ then $\gamma^5\Psi$ is the corresponding eigenspinor of $-i\not{D}(e,A)$.

The equation (4.6) becomes much more interesting if one is able to view the operator $\not{D}(\tilde{e},\widetilde{V})$ as the Dirac operator on the manifold $\widetilde{\mathcal{M}}$ but one must then carefully consider the space on which the operators $\not{D}(e,A)$ and $\not{D}(\tilde{e},\widetilde{V})$ act. The reason being that typically one is interesting in computing the index of the Dirac operator as a trace over states

$$\text{index}\,\not{D}\;=\;\text{Tr}\left[\gamma^5\,\exp\left(\frac{\beta}{2}\not{D}^{\,2}\right)\right], \tag{4.7}$$

where the trace extends over the space of spinors living on the manifold of interest. However, we do not want to compute the index of the second Dirac operator $\not{D}(\tilde{e},\widetilde{V})$ on the manifold $\mathcal{M}$, but rather on the new manifold $\widetilde{\mathcal{M}}$ obtained by expressing $\not{D}(\tilde{e},\widetilde{V})=\gamma^r\,\tilde{e}_r{}^\mu\,\partial_\mu+\cdots$ and regarding $\tilde{e}_r{}^\mu$ as the inverse vierbein. Only in this way do we obtain a relation between index theorems on independent manifolds $\mathcal{M}$ and $\widetilde{\mathcal{M}}$. *A priori*, however, there is no reason to expect that the space of spinors on $\mathcal{M}$ should coincide with that on $\widetilde{\mathcal{M}}$. Furthermore, although the operator $\not{D}(\tilde{e},\widetilde{V})$ was assumed to be antihermitean on the manifold $\mathcal{M}$, since the volume elements of the two manifolds will in general not be equal, the operator $\not{D}(\tilde{e},\widetilde{V})$ will have no definite hermiticity viewed as an operator on spinors on $\widetilde{\mathcal{M}}$. Therefore we make the additional assumption, (which holds for the example we have in mind):

[6]The above argument was pointed out to us by Jan-Willem van Holten [4].

Assumption: The space of spinors (which were defined in detail in subsection 3.1 in such a way that index problems are well-posed) viewed as a four-component space of functions on $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ coincide.

Notice that as a consequence the index problem for the operator $\not{D}(\tilde{e},\widetilde{V})$ is well posed on the manifold $\widetilde{\mathcal{M}}$. This can be seen as follows. Even though the operator $\not{D}(\tilde{e},\widetilde{V})$ has no definite hermiticity on the manifold $\widetilde{\mathcal{M}}$ the operator $\widetilde{\not{D}}(\tilde{e},\widetilde{V}) \equiv \sqrt{g(e)/g(\tilde{e})}\,\not{D}(\tilde{e},\widetilde{V})\,\sqrt{g(\tilde{e})/g(e)}$ is anti-hermitean on $\widetilde{\mathcal{M}}$. Furthermore if Ψ_λ are a complete orthonormal set of eigenfunctions of $\not{D}(\tilde{e},\widetilde{V})$ on $\mathcal{M}$ then $\sqrt{g(e)/g(\tilde{e})}\Psi_\lambda$ are a complete set of orthonormal eigenfunctions of $\widetilde{\not{D}}(\tilde{e},\widetilde{V})$ on $\widetilde{\mathcal{M}}$. Clearly however, by the assumption, if the above similarity transformation is non-singular then the set of functions Ψ_λ are still complete (although no longer orthonormal) on $\widetilde{\mathcal{M}}$ so that $\not{D}(\tilde{e},\widetilde{V})$ can still be diagonalised on $\widetilde{\mathcal{M}}$ leading to a well posed index problem on that manifold.

It should now be clear that given the above assumption we have equality of well posed index problems calculated on $\mathcal{M}$ and $\widetilde{\mathcal{M}}$,

$$\begin{aligned}\text{index}\,\not{D}(e,A) &\equiv \mathrm{Tr}_{\mathcal{M}}\,\gamma^5 \exp\left(\frac{\beta}{2}\not{D}(e,A)^2\right)\\ &= \mathrm{Tr}_{\widetilde{\mathcal{M}}}\,\gamma^5 \exp\left(\frac{\beta}{2}\not{D}(\tilde{e},\widetilde{V})^2\right)\\ &\equiv \text{index}\,\not{D}\left(\tilde{e},\widetilde{V}\right),\end{aligned} \tag{4.8}$$

which is the main result of this section. If we recall the discussion of boundary conditions for spinors on manifolds with boundary in section 3.1, we see that to verify our assumption for the manifolds $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ one only needs to require equality of the projection operators

$$\sum_{l\geq 0}|l\,\rangle\langle\, l| \;=\; \sum_{\tilde{l}\geq 0}|\,\tilde{l}\,\rangle\langle\,\tilde{l}\,|\,, \tag{4.9}$$

where $|l\,\rangle$ and $|\,\tilde{l}\,\rangle$ are the eigenvalues of the respective boundary Dirac operators B and $\widetilde{B}$. Remarkably, we find that (4.9) holds for the Taub-NUT and dual Taub-NUT manifolds.

Note that to compute the index of $\not{D}(\tilde{e},\widetilde{V})$ on $\widetilde{\mathcal{M}}$ we can split the operator into a sum of antihermitean and hermitean operators on $\widetilde{\mathcal{M}}$ by writing

$$\not{D}(\tilde{e},\widetilde{V}) \;\equiv\; \not{D}(\tilde{e},0) \;+\; \gamma^\mu \widetilde{V}_\mu\,, \tag{4.10}$$

with

$$\not{D}(\tilde{e},0) = \left[\not{D}(\tilde{e},\widetilde{V}) - \frac{1}{2}\gamma^\mu(\omega^\rho{}_{\rho\mu} - \omega(\tilde{e})^\rho{}_{\rho\mu})\right]\,, \tag{4.11}$$

$$\widetilde{V}_\mu = \frac{1}{2}\,\omega^\rho{}_{\rho\mu} - \frac{1}{2}\,\omega(\tilde{e})^\rho{}_{\rho\mu}\,, \tag{4.12}$$

where $\omega(\tilde{e})_{\mu rs}$ is the torsion free part of the spin connection computed from the inverse vierbein $\tilde{e}_r{}^\mu$. The trick now is that the hermitean term $\gamma^\mu \widetilde{V}_\mu$ is nothing but the coupling of the Dirac operator to a purely imaginary abelian gauge field whose contribution to the index we may obtain by a naive analytic continuation from the well known results for the index of the Dirac operator with such couplings.[7]

In the remainder of this section we show that manifolds with mutually anti-commuting Dirac operators can be found if invertible Killing-Yano tensor exist and the above relation is realised explicitly by Taub-NUT and its dual manifold. Also provided are all relevant geometric data for these manifolds.

4.2 Killing-Yano dual manifolds

Manifolds with torsion admitting two "Dirac operators" that satisfy the properties discussed in the previous section do in fact exist and are understood in a systematic way [5] (at least in the case where the Dirac operator is viewed as a classical supercharge). Before we move on to an explicit index computation we describe the Killing-Yano technique for generating these manifolds.

The operator $\not{D}(\tilde{e},\widetilde{V})$ can be viewed as the quantum analogue of the supercharge of a spinning particle and in this context the additional Dirac operator $\not{D}(e,A)$ generates the extended supersymmetry. Up to ordering ambiguities this means that the conditions on the tensors $e_\mu{}^r$ and A_{rst} appearing in the additional Dirac operator such that it anticommutes with the original Dirac operator can be deduced from the existence conditions of extended supersymmetry of the classical model [39] (see [5] for the extension to include torsion). In short, additional supercharges on the manifold $\widetilde{\mathcal{M}}$ are given by

$$\not{D}(e,A) = \gamma^r e_r{}^\mu \left(\partial_\mu + \frac{1}{4}\left(\omega(\tilde{e})_{\mu st} - \frac{1}{3}(e^{-1})_\mu{}^p c_{pst}\right)\gamma^{st}\right), \qquad (4.13)$$

where the tensor $e_r{}^\mu$ satisfies

$$D(\tilde{e})_\mu e_\nu{}^r + D(\tilde{e})_\nu e_\mu{}^r = 0\,, \qquad (4.14)$$

which is the Killing-Yano [40] equation for $e_\nu{}^r$ on the manifold $\widetilde{\mathcal{M}}$. Here we have raised and lowered indices with the vierbein $\tilde{e}_\mu{}^r$ on $\widetilde{\mathcal{M}}$ so that $e_\nu{}^r = \tilde{e}_\nu{}^s e_s{}^\mu \tilde{e}_\mu{}^r$ and $D(\tilde{e})_\mu$ denotes the torsion-free covariant derivative on $\widetilde{\mathcal{M}}$. The connection is determined by requiring

$$c_{rst} = D(\tilde{e})_{[r} e_{st]}\,, \qquad (4.15)$$

[7] In the example below we will anyway find a vanishing contribution from this additional abelian vector coupling, however in general it is quite simple to handle. The bulk term is just the usual result for the abelian anomaly modulo an overall minus sign due to the analytic continuation. The Chern-Simons and APS boundary corrections are computed exactly as described in section 3.

where again we have employed the vierbeine on $\widetilde{\mathcal{M}}$ to flatten indices. Note however that when we come to regard $e_r{}^\mu$ as the inverse vierbein on the dual manifold $\mathcal{M}$ we will denote its inverse (i.e. the vierbein itself on $\mathcal{M}$) as $e_\mu{}^r \equiv (e^{-1})_\mu{}^r$. Viewing $\omega(\tilde{e})_{\mu st} - \frac{1}{3}(e^{-1})_\mu{}^p c_{pst}$ as the connection on the dual manifold $\mathcal{M}$ the contortion on that manifold is then

$$K_{t\mu s} = \omega(\tilde{e})_{\mu st} - \omega(e)_{\mu st} - \frac{1}{3}(e^{-1})_\mu{}^p c_{pst}\,. \tag{4.16}$$

the totally antisymmetric part (on $\mathcal{M}$) of which defines the tensor $A_{rst} = K_{[rst]}$.

Indeed just as complex structures (with one flat index) can be used to define a local Lorentz frame different from the frame spanned by the inverse vierbein, an interpretation of the Killing-Yano tensor $e_r{}^\mu$ as an inverse vierbein is possible as well. In this case, however, the manifold for which the inverse vierbein is $e_\mu{}^r$ is not identical to the original one given by $\tilde{e}_\mu{}^r$ (the square $e_\mu{}^r e_{\nu r}$ is a Killing tensor which does not normally coincide with the metric on $\mathcal{M}$). This observation was first elucidated in [5] and plays a central rôle in this paper by providing an example of manifolds with mutually anticommuting Dirac operators.

At the quantum level one must study the possible orderings when making the transition from a classical supercharge to the Dirac operator acting on a spinorial Hilbert space. For example, we have already discussed this problem in detail in section 2 (there symmetry principles such as general coordinate and local Lorentz invariance of the index are central considerations). The key observation of the previous subsection in this respect is the rôle of the coupling to the trace of the torsion $\widetilde{V}^\mu$. Needless to say exactly this coupling is absent in the classical Poisson bracket formulation of the Killing-Yano technique. Upon inclusion of the torsion trace one must reanalyse the quantum anticommutator of Dirac operators. In explicit examples we have found that in order to obtain a vanishing anticommutator one must ensure that both $\not{D}(\tilde{e}, \widetilde{V})$ and $\not{D}(e, A)$ are antihermitean on the manifold in question which necessitates the addition of the abovementioned torsion trace $\widetilde{V}^\mu$.

Unfortunately, manifolds which admit Killing-Yano tensors are not nearly as well understood as their cousins that play a role in $D \geq 2$ extended supersymmetry: Kähler and hyper-Kähler manifolds. The most extensive systematic study so far was made in [41, 42]. Though very explicit, their analysis only concerns the local geometrical structure of manifolds admitting Killing-Yano tensors. Lacking a systematic topological analysis, this information has to be extracted for every example being studied. For this reason, we will in this paper focus on the Taub-NUT manifold [43, 44] for all sample calculations. It should however be noted that at least the Kerr metric is tractable as well; an extension of those results to include the Kerr-Newman family could be used to study a generalisation of our results to include electromagnetic coupling, although there are no conceptual problems expected there. The much simpler Schwarzschild metric only admits a *non-invertible* Killing-Yano tensor, which therefore does not lead to a non-singular inverse vierbein.

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4.3 Example: Taub-NUT geometry and its dual

Both Taub-NUT and dual Taub-NUT possess an SO(3) isometry generated by

$$
\begin{aligned}
k_1{}^\mu \partial_\mu = k_1 &= -\sin\phi\partial_{[\theta]} - \cos\phi\cot\theta\partial_{[\phi]} + \frac{\cos\phi}{\sin\theta}\partial_{[\psi]}\,, \\
k_2 &= \cos\phi\partial_{[\theta]} - \sin\phi\cot\theta\partial_{[\phi]} + \frac{\sin\phi}{\sin\theta}\partial_{[\psi]}\,,
\end{aligned}
\tag{4.17}
$$

as well as the trivial vector

$$
k_3 = \partial_{[\phi]}\,. \tag{4.18}
$$

(there is also a fourth trivial Killing vector $k_4 = \partial_{[\psi]}$). For the computation of the bulk contributions to the index, as reported in section 4.4.1, it is not very important to make this isometry manifest and one can use an (almost) diagonal inverse vierbein based on the metrics (1.2) or (1.3). It is, however, crucial to make this symmetry manifest in order to use the η invariant computations for manifolds with S^3 boundary of subsection 3.2. To achieve this one makes use of the left-invariant forms Σ^i. These forms satisfy

$$
\mathcal{L}_k \Sigma^i = \big(\iota_k \mathrm{d} + \mathrm{d}\iota_k\big)\,\Sigma^i = \big(k^\mu\partial_\mu\Sigma^i_\nu + \partial_\nu k^\mu \Sigma^i_\mu\big)\,\mathrm{d}x^\nu = 0\,, \tag{4.19}
$$

where $\mathcal{L}_k$ is the Lie derivative with respect to any one of the three Killing vector fields k_i. Explicitly,

$$
\begin{aligned}
\Sigma_\mu{}^1\,\mathrm{d}x^\mu = \Sigma^1 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\phi\,, \\
\Sigma^2 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\phi\,, \\
\Sigma^3 &= \cos\theta\,\mathrm{d}\phi + \mathrm{d}\psi\,,
\end{aligned}
\tag{4.20}
$$

and they satisfy $\mathrm{d}\Sigma^k = -(1/2)\epsilon^{ijk}\Sigma^i\wedge\Sigma^j$. The metric (1.2) of Taub-NUT then becomes

$$
\mathrm{d}\tilde{s}^2 = \frac{r+2m}{r}\,\mathrm{d}r^2 + r(r+2m)\,\left[(\Sigma^1)^2 + (\Sigma^2)^2\right] + \frac{4rm^2}{r+2m}\,\left[\Sigma^3\right]^2, \tag{4.21}
$$

whose determinant is given by

$$
g(\tilde{e}) = \Big[\,2mr(r+2m)\sin\theta\,\Big]^2. \tag{4.22}
$$

The Killing-Yano tensor can also be expressed using the invariant one-forms,

$$
\widetilde{Y} \equiv \tilde{e}_\mu{}^r e_r{}^\rho g(\tilde{e})_{\rho\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = -2m\,\mathrm{d}r\wedge\Sigma^3 + \frac{1}{m}(r+2m)(r+m)\,\Sigma^1\wedge\Sigma^2\,, \tag{4.23}
$$

and is therefore SO(3) invariant as well. The dual Taub-NUT manifold, obtained through the procedure sketched in the present section, is given by the metric

$$
\mathrm{d}s^2 = \frac{r+2m}{r}\,\mathrm{d}r^2 + \frac{r(r+2m)}{S^2}\,\left[(\Sigma^1)^2 + (\Sigma^2)^2\right] + \frac{4rm^2}{r+2m}\,\left[\Sigma^3\right]^2. \tag{4.24}
$$

$$\tilde{e}^1 = \sqrt{\frac{r+2m}{r}}\,\mathrm{d}r\,, \qquad \tilde{e}^2 = \sqrt{r^2+2mr}\,\Sigma^1\,,$$
$$\tilde{e}^3 = \sqrt{r^2+2mr}\,\Sigma^2\,, \quad \tilde{e}^4 = \sqrt{\frac{4rm^2}{r+2m}}\,\Sigma^3\,.$$

$$\begin{aligned}
\widetilde{\omega}_{12} &= -\frac{r+m}{r+2m}\,\Sigma^1\,, & \widetilde{\omega}_{13} &= -\frac{r+m}{r+2m}\,\Sigma^2\,,\\
\widetilde{\omega}_{14} &= -2\frac{m^2}{(r+2m)^2}\,\Sigma^3\,, & \widetilde{\omega}_{23} &= -\frac{r^2+2m^2+4rm}{(r+2m)^2}\,\Sigma^3\,,\\
\widetilde{\omega}_{24} &= \frac{m}{r+2m}\,\Sigma^2\,, & \widetilde{\omega}_{34} &= -\frac{m}{r+2m}\,\Sigma^1\,.
\end{aligned}$$

$$\begin{aligned}
-\widetilde{R}_{43} = \widetilde{R}_{21} &= \frac{m}{(r+2m)^2}\,\mathrm{d}r\wedge\Sigma^1 - \frac{2rm^2}{(r+2m)^3}\,\Sigma^2\wedge\Sigma^3\,,\\
\widetilde{R}_{42} = \widetilde{R}_{31} &= \frac{m}{(r+2m)^2}\,\mathrm{d}r\wedge\Sigma^2 + \frac{2rm^2}{(r+2m)^3}\,\Sigma^1\wedge\Sigma^3\,,\\
-\widetilde{R}_{32} = \widetilde{R}_{41} &= -\frac{4m^2}{(r+2m)^3}\,\mathrm{d}r\wedge\Sigma^3 + \frac{2rm}{(r+2m)^2}\,\Sigma^1\wedge\Sigma^2\,.
\end{aligned}$$

Table 6: Geometrical data of the Taub-NUT manifold.

with determinant

$$g(e) \;=\; \Big[\,\frac{2mr(r+2m)\sin\theta}{S^2}\,\Big]^2\,, \tag{4.25}$$

where $S = (r+m)/m$.

The connections and curvatures are all summarised in the two tables 6 and 7. Before we discuss the Dirac operators, let us make a few remarks about the displayed results. First of all, the curvature of the dual manifold is not (anti)self-dual, in contrast to Taub-NUT (note that in our conventions $\widetilde{R}^{rs} = -(1/2)\epsilon^{rtsu}\widetilde{R}_{su}$). Its asymptotic geometry is also different: whereas Taub-NUT is asymptotically a flat metric, the dual metric has a volume element independent of r at large radius; it tends to an infinite tube.

Of course, we must also specify the coordinate ranges for the angular coordinates θ, ϕ and ψ. It is easy to see that for both Taub-NUT and dual Taub-NUT one has $\theta \in [0,\pi)$ and $\phi \in [0,2\pi)$. For the coordinate ψ one finds $\psi \in [0,4\pi)$ by ensuring the absence of the NUT singularity at $r=0$. In detail, one can write both the Taub-NUT and dual Taub-NUT metrics on a constant $(\theta\phi)$-slice in the vicinity of $r=0$ as $\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{1}{4}x^2\mathrm{d}\psi^2$ via the coordinate transformation $x^2 = 8mr$. Requiring this be the metric of a two-dimensional plane in polar coordinates we find the quoted result for ψ.

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$$
\begin{aligned}
e^1 &= \sqrt{\frac{4rm^2}{r+2m}}\,\Sigma^3\,, & e^2 &= -\frac{\sqrt{m^2 r(r+2m)}}{r+m}\,\Sigma^2\,,\\
e^3 &= \frac{\sqrt{m^2 r(r+2m)}}{r+m}\,\Sigma^1\,, & e^4 &= -\sqrt{\frac{r+2m}{r}}\,\mathrm{d}r\,.
\end{aligned}
$$

$$
\begin{aligned}
\omega_{12} &= -\frac{r+m}{r+2m}\,\Sigma^1\,, & \omega_{13} &= -\frac{r+m}{r+2m}\,\Sigma^2\,, & A_{12} &= \frac{1}{3}\frac{r}{r+m}\,\Sigma^1\,,\\
\omega_{14} &= -2\frac{m^2}{(r+2m)^2}\,\Sigma^3\,, & \omega_{23} &= \frac{r^2-2m^2}{(r+2m)^2}\,\Sigma^3\,, & A_{13} &= \frac{1}{3}\frac{r}{r+m}\,\Sigma^2\,,\\
\omega_{24} &= \frac{m^3}{(r+2m)(r+m)^2}\,\Sigma^2\,, & \omega_{34} &= -\frac{m^3}{(r+2m)(r+m)^2}\,\Sigma^1\,, & A_{23} &= \frac{2}{3}\frac{r}{r+2m}\,\Sigma^3\,.
\end{aligned}
$$

$$
\begin{aligned}
R_{21} &= \frac{m}{(r+2m)^2}\,\mathrm{d}r\wedge\Sigma^1 - 2\left[\frac{(r+m)^3}{(r+2m)^3} - \frac{m^5}{(r+m)^2(r+2m)^3}\right]\Sigma^2\wedge\Sigma^3\,,\\
R_{31} &= \frac{m}{(r+2m)^2}\,\mathrm{d}r\wedge\Sigma^2 + 2\left[\frac{(r+m)^3}{(r+2m)^3} - \frac{m^5}{(r+m)^2(r+2m)^3}\right]\Sigma^1\wedge\Sigma^3\,,\\
R_{41} &= -\frac{4m^2}{(r+2m)^3}\,\mathrm{d}r\wedge\Sigma^3 - \frac{2m^2 r}{(r+m)(r+2m)^2}\,\Sigma^1\wedge\Sigma^2\,,
\end{aligned}
$$

$$
\begin{aligned}
R_{43} &= -\frac{m^3(3r+5m)}{(r+2m)^2(r+m)^3}\,\mathrm{d}r\wedge\Sigma^1 + \frac{2rm^2}{(r+2m)^3}\,\Sigma^2\wedge\Sigma^3\\
R_{42} &= \frac{m^3(3r+5m)}{(r+2m)^2(r+m)^3}\,\mathrm{d}r\wedge\Sigma^2 + \frac{2rm^2}{(r+2m)^3}\,\Sigma^1\wedge\Sigma^3\,,\\
R_{32} &= -\frac{4m(r+m)}{(r+2m)^3}\,\mathrm{d}r\wedge\Sigma^3 + \left[\frac{2r^2+2rm-m^2}{(r+2m)^2} + \frac{m^6}{(r+2m)^2(r+m)^4}\right]\Sigma^1\wedge\Sigma^2\,.
\end{aligned}
$$

Table 7: Geometrical data of the dual Taub-NUT manifold.

For the Dirac operators we actually need the inverse of the invariant one-forms. We denote them by Σ_i just as in subsection 3.2. They are the vectors commuting with the Killing vectors,

$$
[k_i, \Sigma_j] = 0\,, \tag{4.26}
$$

and in components one finds

$$
\begin{aligned}
{\Sigma_1}^\mu\partial_\mu = \Sigma_1 &= \cos\psi\,\partial_\theta + \frac{\sin\psi}{\sin\theta}\,\partial_\phi - \frac{\cos\theta\sin\psi}{\sin\theta}\,\partial_\psi\,,\\
\Sigma_2 &= -\sin\psi\,\partial_\theta + \frac{\cos\psi}{\sin\theta}\,\partial_\phi - \frac{\cos\psi\cos\theta}{\sin\theta}\,\partial_\psi\,,\\
\Sigma_3 &= \partial_\psi\,.
\end{aligned} \tag{4.27}
$$

In the manifestly $SO(3)$ invariant basis the Dirac operator on dual Taub-NUT with Killing-Yano torsion is given by

$$\not{D}(e,A) = \frac{1}{rV}\gamma^4\left[r\partial_r + \frac{1}{2}\left(\frac{2r+3m}{r+2m} - \frac{2r}{(r+m)}\right) + \right.$$
$$\left. + \left\{S\left(\gamma^{34}\Sigma_1 + \gamma^{42}\Sigma_2\right) + \frac{1}{\lambda}\gamma^{14}\Sigma_3 - \gamma^{1234}\frac{\lambda^2+2}{4\lambda}\right\}\right]. \quad (4.28)$$

It is antihermitean on the dual Taub-NUT metric and anticommutes with the operator

$$\not{D}(\tilde{e},\widetilde{V}) = \frac{1}{rV}\gamma^1\left[r\partial_r + \frac{1}{2}\left(\frac{2r+3m}{r+2m} - \frac{2r}{(r+m)}\right) + \right.$$
$$\left. + \left\{\gamma^{12}\Sigma_1 + \gamma^{13}\Sigma_2 + \frac{1}{\lambda}\gamma^{14}\Sigma_3 - \gamma^{1234}\frac{\lambda^2+2}{4\lambda}\right\}\right]. \quad (4.29)$$

The operator $\not{D}(\tilde{e},\widetilde{V})$ is antihermitean on the dual Taub-NUT manifold but has no definite hermiticity on Taub-NUT itself where it can be regarded as the Dirac operator coupled to the trace of the torsion $\gamma^\mu\widetilde{V}_\mu = -\gamma^1/[(r+m)V]$, or in other words a purely imaginary abelian vector field. In fact the replacement

$$\partial_r \to \partial_r + \frac{1}{r(r+m)} \quad (4.30)$$

renders the operators $\not{D}(\tilde{e},\widetilde{V})$ and $\not{D}(e,A)$ antihermitean on Taub-NUT and in the case of the former one obtains $\not{D}(e)$, the torsion-free Taub-NUT Dirac operator. Note that in the above $\lambda = 2m/(r+2m)$, $S = (r+m)/m$ and $V = \sqrt{(r+2m)/r}$.

4.4 Example: Index theorems for Taub-NUT and dual Taub-NUT

4.4.1 Bulk

Using the geometrical data of the two manifolds we just presented, all terms in (3.31) can be computed and the indices can be compared. Let us start with the bulk part. For the dual Taub-NUT manifold there are contributions from all terms in expression (2.19) except for the $F\wedge F$ term since $F = \mathrm{d}A$ vanishes exactly for this example. Putting the boundary at a finite radius $r = r_b$ we obtain the (rather unenlightening) result

$$\frac{1}{24.8\pi^2}\int_{\mathcal{M}}\left[R_{mn}\wedge R^{nm} - \frac{1}{2}F\wedge F - 2\sqrt{g}\,D_\mu\mathcal{K}^\mu\right] = \quad (4.31)$$
$$= \frac{1}{12}\frac{r_b^2\left(r_b^6 + 12mr_b^5 + 86r_b^4m^2 + 340m^3r_b^3 + 753r_b^2m^4 + 872r_bm^5 + 408m^6\right)}{(r_b+2m)^4(r_b+m)^4}.$$

In the limit $r_b \to \infty$ this yields the number $1/12$. If one turns the torsion off, the bulk result for dual Taub-NUT is then simply

$$\frac{1}{24.8\pi^2}\int_{\mathcal{M}} R_{mn}\wedge R^{nm} = \frac{1}{12}\frac{r_b^2\left(9r_b^4 + 42r_b^3m + 73r_b^2m^2 + 56r_bm^3 + 24m^4\right)}{(r_b+2m)^4(r_b+m)^2}. \quad (4.32)$$

The analogous computation for Taub-NUT is well known [45, 46],

$$\frac{1}{24.8\pi^2}\int_{\widetilde{\mathcal{M}}} R_{mn}\wedge R^{nm} = \frac{1}{12}\frac{r_b^2\,(r_b^2+8r_bm+24m^2)}{(r_b+2m)^4}\,. \tag{4.33}$$

For a boundary at infinity, this produces the answer $1/12$ (or equivalently a Pontrjagin number 2). For Taub-NUT it is known that this number, which seems to lead to a fractional index, in fact gets modified by the boundary contributions [12, 13, 14]. Because of the general results discussed in the first part of this section, we expect that a similar correction will arise for the dual Taub-NUT manifold as well.

4.4.2 η invariant

The boundaries of Taub-NUT and the dual Taub-NUT are S^3 so our general discussion for the computation of the η invariant in subsection 3.2 applies. From (4.28) we obtain the dual Taub-NUT boundary operator

$$\mathcal{B} = \begin{pmatrix} P(S,Z) & 0 \\ 0 & -P(-S,Z)\end{pmatrix}, \tag{4.34}$$

analysed above in 3.2 with

$$\lambda = \frac{2m}{r+2m}\,, \qquad S = \frac{r+m}{m}\,, \tag{4.35}$$

as well as

$$Z = \begin{cases} 0 & \text{Killing-Yano torsion,} \\ \dfrac{r}{2m} & \text{torsion-free.}\end{cases} \tag{4.36}$$

(the torsion-free case is in fact the value for Z at which the η invariant can be obtained from the one of Taub-NUT by replacing $\lambda \to S\lambda$, but we have presented the general formula (3.25) just in case one wants to study the interpolation between the two cases). We then find that $\eta(0)$ with the boundary at finite radius $r = r_b$ for dual Taub-NUT with Killing-Yano torsion is

$$\frac{1}{2}\eta_B(0) = -\frac{1}{12}\frac{r_b(r_b^5+10mr_b^4+65r_b^3m^2+168m^3r_b^2+168r_bm^4+48m^5)}{(r_b+2m)^4(r_b+m)^2} \tag{4.37}$$

and for dual Taub-NUT without torsion

$$\frac{1}{2}\eta_{B\,(A=0)}(0) = -\frac{1}{12}\frac{(3r_b+4m)^2r_b^2}{(r_b+2m)^4}\,. \tag{4.38}$$

For Taub-NUT itself, from (4.29) the boundary Dirac operator is

$$\widetilde{\mathcal{B}} = \begin{pmatrix} P(1,0) & 0 \\ 0 & -P(1,0)\end{pmatrix}. \tag{4.39}$$

Again $\lambda = 2m/(r+2m)$ where this result as well as (4.34) are given in the same chiral basis (3.10) to facilitate comparison of the space of boundary solutions with positive eigenvalues in subsection 4.4.4 below. Clearly the analysis of subsection 3.2 applies equally well to the Taub-NUT boundary Dirac operator (4.39) and importantly the solutions still take the form quoted in table 5. Therefore, for the Taub-NUT manifold, we reproduce the results of [46] and find

$$\frac{1}{2}\eta_{\widetilde{B}}(0) = -\frac{1}{12}\frac{r_b^2(4m+r_b)^2}{(r_b+2m)^4}\,. \tag{4.40}$$

Note that the expressions (4.40) and (4.37) go to $-1/12$ when the boundary is moved out to infinite radius, $r_b \to \infty$. On the other hand, the limiting value for $\frac{1}{2}\eta(0)$ for dual Taub-NUT without torsion is $-3/4$.

Finally we note that the number of harmonic boundary spinors h vanishes for Taub-NUT and dual Taub-NUT with or without torsion.

4.4.3 Chern-Simons correction

For the dual of Taub-NUT, we have computed the difference between the boundary integral at infinity for the product metric and the same integral for the full metric to be (the contribution $A\mathrm{d}A$ from the abelian axial vector field drops out since in this example $F = \mathrm{d}A \equiv 0$)

$$\begin{aligned}\frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}}\Big[\omega^\sharp\wedge R^\sharp - \frac{1}{3}\omega^\sharp\wedge\omega^\sharp\wedge\omega^\sharp - 2\,\mathcal{K}^\sharp - \omega\wedge R + \frac{1}{3}\omega\wedge\omega\wedge\omega + 2\,\mathcal{K}\Big] &= \\ = \frac{2}{3}\frac{r_b m^3(-4r_b^4 - 23r_b^3 m - 40m^2r_b^2 - 18r_b m^3 + 6m^4)}{(r_b+2m)^4(r_b+m)^4}\,. \end{aligned}\tag{4.41}$$

Since the dual metric is already a product metric at infinity, one expects this correction to vanish in the large r_b limit, which is indeed the case. Turning off the torsion this reduces to

$$\begin{aligned}\frac{1}{24.8\pi^2}\int_{\partial\mathcal{M}}\Big[\omega^\sharp\wedge R^\sharp - \frac{1}{3}\omega^\sharp\wedge\omega^\sharp\wedge\omega^\sharp - \omega\wedge R + \frac{1}{3}\omega\wedge\omega\wedge\omega\Big] &= \\ = -\frac{2}{3}\frac{m^4 r_b^2}{(r_b+2m)^4(r_b+m)^2}\,. \end{aligned}\tag{4.42}$$

Finally, for Taub-NUT itself, we confirm the result of [46] which in our coordinate system is

$$\frac{1}{24.8\pi^2}\int_{\partial\widetilde{\mathcal{M}}}\Big[\omega^\sharp\wedge R^\sharp - \frac{1}{3}\omega^\sharp\wedge\omega^\sharp\wedge\omega^\sharp - \omega\wedge R + \frac{1}{3}\omega\wedge\omega\wedge\omega\Big] = -\frac{2}{3}\frac{m^2 r_b^2}{(r_b+2m)^4}\,. \tag{4.43}$$

The above expression also vanishes in the large r_b limit.

4.4.4 Equality of indices

Although it is now easy to verify the equality of the indices on the Taub-NUT and dual Taub-NUT manifolds, it would be remiss of us to do so without first checking the validity the assumption of subsection 4.1 above. Therefore we return to the solutions presented in table 5 and check that the eigenspaces generated by the solutions with positive eigenvalues coincide for Taub-NUT and dual Taub-NUT. Clearly the eigenvalues (setting $Z = 0$, since $Z = r/(2m) \neq 0$ corresponds to dual Taub-NUT without torsion) $p/2\lambda + \lambda/4$ and $\lambda/4 + \sqrt{4pq(\lambda S)^2 + (p-q)^2}/2\lambda$ ($S = 1$ for Taub-NUT and $S = (r+m)/m$ for dual Taub-NUT) are always positive. Furthermore, from the form of the explicit solutions, we see that the eigenspaces generated by the corresponding eigenvectors are identical for both manifolds. (Strictly, one must take r_b sufficiently small for the preceding statement to hold. However, this is sufficient since, as we are about to show, neither manifold possesses harmonic boundary spinors at any positive r_b so that for both manifolds, the index is independent of $r_b > 0$.)

If the solutions corresponding to the negative root $\lambda/4 - \sqrt{4pq(\lambda S)^2 + (p-q)^2}/2\lambda$ were all to have strictly negative eigenvalues clearly we would be done since the projective boundary conditions then coincide for the two manifolds. Yet for any non-negative radius r to the boundary, the eigenvalue with the negative root can never be positive for either manifold. To see this we just need to look at the minimum of the square root $\sqrt{4pq(\lambda S)^2 + (p-q)^2}$ which occurs at $p = 1 = q$. But for both manifolds $\lambda/4 - S$ is negative so long as r is non-negative which completes our verification of the assumption.

We are now justified in comparing the indices of Taub-NUT and dual Taub-NUT. Therefore, adding the bulk contribution (4.31), the η invariant (4.37) and the Chern-Simons correction (4.41) of the dual Taub-NUT manifold, we find that the r_b dependence drops out completely,

$$
\begin{aligned}
&\mathrm{index}\,\not{D}\,(e,A) = \qquad\qquad (4.44)\\
&= \frac{r_b^8 + 12r_b^7 m + 86 r_b^6 m^2 + 340 r_b^5 m^3 + 753 r_b^4 m^4 + 872 r_b^3 m^5 + 408 r_b^2 m^6}{12(r_b+2m)^4 (r_b+m)^4} - \\
&- \frac{r_b^8 + 12r_b^7 m + 86 r_b^6 m^2 + 308 r_b^5 m^3 + 569 r_b^4 m^4 + 552 r_b^3 m^5 + 264 r_b^2 m^6 + 48 r_b m^7}{12(r_b+2m)^4 (r_b+m)^4} - \\
&\qquad - \frac{32 r_b^5 m^3 + 184 r_b^4 m^4 + 320 r_b^3 m^5 + 144 r_b^2 m^6 - 48 r_b m^7}{12(r_b+2m)^4 (r_b+m)^4} = 0\,,
\end{aligned}
$$

(the η invariant has to be added rather than subtracted because our conventions for the gamma matrices imply that the upper component states have negative chirality). This result matches precisely the (well known) result for Taub-NUT obtained by adding (4.33), (4.40) and (4.43) (it is easy to verify that there is no additional contribution at any value of r_b to the Taub-NUT index from the abelian coupling

V^μ) and therefore

$$\text{index} \not{D}\,(e, A) = 0 = \text{index} \not{D}\,(\tilde{e}, \widetilde{V})\,. \tag{4.45}$$

It is interesting to also compute the index for dual Taub-NUT without torsion. In that case, adding (4.32), (4.38) and (4.42), we again find a vanishing result

$$\text{index}\,\not{D}\,(e, A = 0) = 0 \tag{4.46}$$

at any radius to the boundary r_b (one might have even argued that since the dual metric can be smoothly deformed into that of Taub-NUT, this was to be expected, although in the presence of boundary this statement should be carefully re-examined).

Even though the bulk and boundary terms decouple completely in the limit $r_b \to \infty$ for both manifolds, this is not true for general values of r_b indicating the nontrivial nature of our result. Moreover, the torsion terms in the various contributions yield a nontrivial answer which matches precisely with that of the metric terms to produce the r_b independent result (4.45). This concludes our check of the APS index theorem generalised to manifolds with torsion.

5. Summary and conclusions

In this paper we have presented the modifications of the bulk and boundary terms in the index theorems for manifolds with boundary when torsion is added. In addition we have shown how the index theorem for such manifolds can be related when they possess mutually anticommuting Dirac operators. All these results can be explicitly verified for the case of the Taub-NUT and dual Taub-NUT manifolds and we find that the index for both these manifolds vanishes.

Many important technical and physical issues were solved *en route* to the above results. In particular the appearance of the Nieh-Yan tensor at order $\mathcal{O}(\beta^{-1})$ in the bulk index computation has previously caused some controversy [15, 16, 17, 18, 19, 20]. However, once one studies well-posed index problems by carefully imposing boundary terms for spinors, the index should be independent of the inverse regulator $(\text{mass})^2\ \beta$. Therefore one suspects that keeping all regulator dependent terms in the APS analysis in the presence of torsion, all $1/\beta$ terms should exactly cancel. The importance of this remark for the axial anomaly in quantum field theory should not go unnoticed.

Interestingly enough, we note that the torsion does not lead to a modification of the index for the dual Taub-Nut manifold, so that one may wonder whether on general grounds torsion should be expected contribute to the index or not. We are not aware of any general argument suggesting that torsion cannot contribute to the index of manifolds *with boundary* but from a physical viewpoint, where each of the bulk, boundary and generalised Chern-Simons corrections are separately of considerable interest, the existence of such an argument would not diminish the importance of our results which we now summarise.

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The relation found between manifolds with anticommuting Dirac operators involves the so-called Killing-Yano duality. In other words, one must must study the motion of spinning particles in curved space and in this paper we have extended existing studies of classical spinning particle dynamics to the quantum case in backgrounds with torsion. The precise path integral quantisation was given in section 2 and the resolution of possible ordering ambiguities for the Dirac operator by invoking anti-hermiticity was discussed in section 4. As mentioned earlier, our work may also be viewed as a stringent check of the precise path integral quantisation scheme utilised in this paper.

The relationship we found between manifolds with mutually anticommuting Dirac operators depended on being able to understand and compare the Hilbert space of spinors on manifolds with boundary. This analysis was made possible by the work of APS and we have shown that it can be equally well applied to the torsion-full case also. In particular we note that in all our analysis it was possible to take any *finite* radius r_b to the boundary. This allows one to study the non-compact limit in which $r_b \to \infty$ which of course has some topical significance in present day studies of anti de Sitter metrics. The agreement we found between indices at finite r_b is a very strong check of our results.

The generalisations to include torsion that we presented of Hitchin's η invariant computation for squashed S^3 metrics and to Gilkey's Chern-Simons non-product metric boundary correction were in principle straightforward but are of course an important step if one is to understand index theorems in the torsion-full case. We note that it would be desirable also to consider cases where the antisymmetric contortion also has non-vanishing components in the direction normal to the boundary. One might expect then that analogous results to ours would also hold. One might also like to proceed in such a case by searching for other manifolds satisfying the Killing-Yano duality relation. To this end we note that the Kerr-Newman metric of a rotating black hole is certainly tractable along the lines presented in this paper, but we reserve this physically interesting metric for further study.

As a final note we also observe that the generalised Chern-Simons form $C = \omega \wedge R - \frac{1}{3}\omega \wedge \omega \wedge \omega - 2\,\mathcal{K}$ found in section 2 (where, presumably one ought include the order $\hbar^{-1}$ Nieh-Yan term along with possible higher $O(\hbar)$ corrections in $\mathcal{K}$) might represent an interesting three dimensional field theory in its own right. But again we leave such developments to the future.

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Nieuwenhuizen for discussions and are deeply indebted to Nigel Hitchin for providing us the derivation of the residues $f\big((s+1)/2\big)$ and $f\big((s+3)/2\big)$ required in the η invariant computation [36]. The calculations reported in this paper made extensive use of the tensor algebra package GRtensor [47] along with the algebraic manipulation program FORM [48].

A. Conventions and general relativity with torsion

We work exclusively with manifolds of Euclidean signature and our Dirac matrices γ^r are hermitean and satisfy $\{\gamma^r, \gamma^s\} = 2\,\delta^{rs}$. Flat (tangent space) indices are denoted by the lower case Roman alphabet whereas curved indices are members of the Greek alphabet. Products of Dirac matrices are denoted as $\gamma^{r_1 \cdots r_n} = \gamma^{[r_1} \cdots \gamma^{r_n]}$ where we (anti) symmetrise with unit weight and $\gamma^5 = \frac{1}{24}\epsilon^{rstu}\gamma_{rstu} = \gamma^{1234} = \gamma^{5\dagger}$. We will often employ differential form notation in which $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$ and flat $SO(4)$ indices are usually suppressed and understood to be traced over. For a useful review of gravity with torsion see [49].

Our convention for the spin connection in the presence of torsion is encapsulated in the vierbein postulate,

$$0 = \partial_\mu e_\nu{}^r - \Omega^\lambda{}_{\mu\nu} e_\lambda{}^r + \omega_\mu{}^r{}_s e_\nu{}^s\,, \tag{A.1}$$

where the contortion $K^\lambda{}_{\mu\nu}$ is given by

$$\Omega^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - K^\lambda{}_{\mu\nu}\,, \quad \omega_\mu{}^m{}_n = \omega(e)_\mu{}^m{}_n - K^m{}_{\mu n}\,. \tag{A.2}$$

In general, this differs from the torsion itself,

$$T^m = \mathrm{d}e^m + \omega_\mu{}^m{}_n \wedge e^n\,, \qquad \text{or} \qquad T^\rho{}_{\mu\nu} = \Omega^\rho{}_{[\mu\nu]} \tag{A.3}$$

the precise relation being

$$K^\rho{}_{\mu\nu} = g^{\rho\lambda}\left(-T_{\lambda\mu\nu} + T_{\mu\nu\lambda} - T_{\nu\lambda\mu}\right)\,. \tag{A.4}$$

The torsion-free spin connection is expressed in terms of the vierbein by

$$\omega_{\mu\, rs}(e) = \frac{1}{2}\big[-\lambda_{rs\mu} + \lambda_{\mu rs} - \lambda_{\mu sr}\big]\,, \qquad \lambda_{\mu\nu r} = 2\,\partial_{[\mu} e_{\nu]r}\,. \tag{A.5}$$

We define the field strength of the contortion by

$$F_{\mu\nu\rho\sigma} = 4\,\partial_{[\mu} K_{\nu\rho\sigma]}\,. \tag{A.6}$$

Our conventions for the Riemann curvature are

$$R(g,T)_{\mu\nu\rho\sigma} = g_{\sigma\lambda}\,\partial_\nu \Omega^\lambda{}_{\mu\rho} + \Omega_{\sigma\nu\lambda}\Omega^\lambda{}_{\mu\rho} - (\mu \leftrightarrow \nu)\,, \tag{A.7}$$

$$R(\omega)_{\mu\nu mn} = \partial_\mu \omega_{\nu mn} + \omega_{\mu mp}\omega_\nu{}^p{}_n - (\mu \leftrightarrow \nu)\,. \tag{A.8}$$

and $R^{mn} = \mathrm{d}\omega^{mn} + \omega^m{}_p \wedge \omega^{pn} = \frac{1}{2} R_{\mu\nu}{}^{mn} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ (the field strength F is similarly normalised to be $F = \mathrm{d}A = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$). Using (A.1) one verifies that

$$R(g,T)_{\mu\nu\rho\sigma} e_m{}^\rho e_n{}^\sigma = R(\omega)_{\mu\nu mn} \,. \tag{A.9}$$

Our sign convention for the Ricci tensor is $R_{\mu\sigma} = R_\mu{}^\rho{}_{\rho\sigma}$.

We will mostly be dealing with torsion that is fully anti-symmetric, which we denote by writing the symbol $A^\rho{}_{\mu\nu}$ instead of $K^\rho{}_{\mu\nu}$. In this case, the torsion equals minus the contortion, so we will not introduce a separate symbol for it. The axial vector obtained by dualising $A_{\rho\mu\nu}$ is denoted A^μ

$$\begin{aligned} A^\mu &= \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho\sigma} A_{\nu\rho\sigma} \,, \qquad & A_{\mu\nu\rho} &= -\frac{1}{6}\sqrt{g}\, \epsilon_{\mu\nu\rho\lambda} A^\lambda \,, \\ A^{\mu\nu\rho} A_{\mu\nu\rho} &= \frac{1}{6} A^\lambda A_\lambda \,, & F_{\mu\nu} &= 2\,\partial_{[\mu} A_{\nu]} \,, \end{aligned} \tag{A.10}$$

from which one obtains

$$F_{\mu\nu\rho\sigma} = -\frac{2}{3} \partial_{[\mu} (\sqrt{g}\, \epsilon_{\nu\rho\sigma]\lambda} A^\lambda) = \frac{1}{6} \sqrt{g}\, \epsilon_{\mu\nu\rho\sigma} D_\lambda A^\lambda \,, \tag{A.11}$$

using (A.6) and the Schouten identity. For the contractions of the Riemann tensor, it is useful to separate the pieces that only involve a Christoffel connection. When the torsion is fully anti-symmetric, this leads to reasonably compact expressions,

$$R(g,A)_{\mu\nu\rho\sigma} = R(g)_{\mu\nu\rho\sigma} + 2\, D_{[\mu} A_{\nu]\rho\sigma} + 2\, A^\lambda{}_{\rho[\mu} A_{\nu]\sigma\lambda} \,, \tag{A.12}$$

$$R(g,A)_{\mu\sigma} = R(g)_{\mu\sigma} + D_\lambda A^\lambda{}_{\mu\sigma} + A^{\lambda\kappa}{}_\mu A_{\lambda\kappa\sigma} \,, \tag{A.13}$$

$$R(g,A) = R(g) + A^{\lambda\kappa\delta} A_{\lambda\kappa\delta} \,. \tag{A.14}$$

D_μ denotes the covariant derivative *without* torsion. Expressed in terms of the torsion pseudo vector we then have (along with analogous formulae for the Ricci and scalar curvatures)

$$\begin{aligned} R(g,A)_{\mu\nu\rho\sigma} =& R(g)_{\mu\nu\rho\sigma} - \frac{1}{3} D_{[\mu} A^\phi \sqrt{g}\, \epsilon_{\nu]\rho\sigma\phi} + \\ &+ \frac{1}{18} \Big[g_{\mu[\rho} A_{\sigma]} A_\nu - g_{\nu[\rho} A_{\sigma]} A_\mu - g_{\mu[\rho} g_{\sigma]\nu} A^\lambda A_\lambda \Big] \,. \end{aligned} \tag{A.15}$$

Finally we note that although the spin connection, in all generality has 24 components which decompose under SO(4) as a $\underline{4}$ (the trace), a $\underline{16}$ (terms with mixed symmetry) and a $\underline{4}$ (the totally antisymmetric piece), the Dirac operator can only couple via the Dirac matrices to the two $\underline{4}$'s which one may interpret as the coupling to abelian vector and axial vector fields $\widetilde{V}_\mu$ and A_μ respectively. Therefore the most general Dirac operator is given by

$$\not{D}\,(e,V,A) = \gamma^r e_r{}^\mu \left(\partial_\mu + \frac{1}{4} \omega(e)_{\mu st} \gamma^{st} + \widetilde{V}_\mu + \frac{1}{4} A_\mu \gamma^5 \right) ,$$

where the spin connection with torsion is then

$$\omega_{\mu rs} = \omega(e)_{\mu rs} + \frac{4}{3} e_{\mu[r} V_{s]} - \frac{1}{6} \sqrt{g(e)}\, \epsilon_{\mu\nu\rho\sigma} e^{\nu}{}_{r} e^{\rho}{}_{s} A^{\sigma}$$

and $\omega(e)_{\mu rs}$ and $g(e)$ are the torsion-less spin connection and metric determinant computed from the vierbein $e_\mu{}^r$. We abbreviate $\not{D}\,(e,V,A)$ to $\not{D}\,(e,\widetilde{V}) = \not{D}\,(e,V,0)$, $\not{D}\,(e,A) = \not{D}\,(e,0,A)$ or even just $\not{D}$, but our intention should always be clear from the context.

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